CHAPTER

2

Analytic Geometry

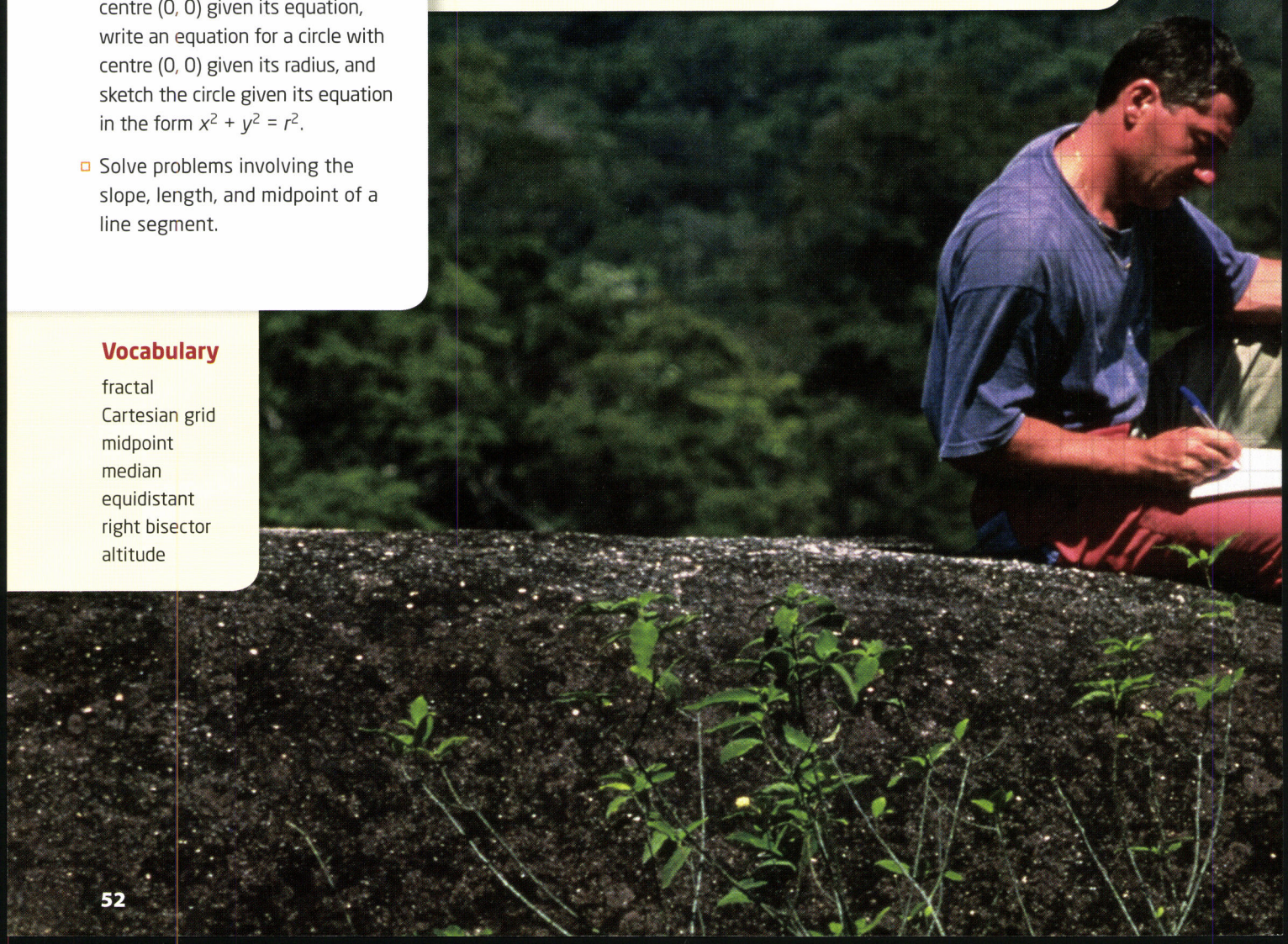
A global positioning system (GPS) device uses signals from satellites to calculate its location. The GPS describes locations with three coordinates: latitude, longitude, and altitude. Coordinate systems are the basis of analytic geometry, the algebraic analysis of geometric shapes. This chapter uses coordinates to calculate distances, find halfway points, and analyse various shapes. The chapter also shows how to apply these skills to solve problems in design and construction.

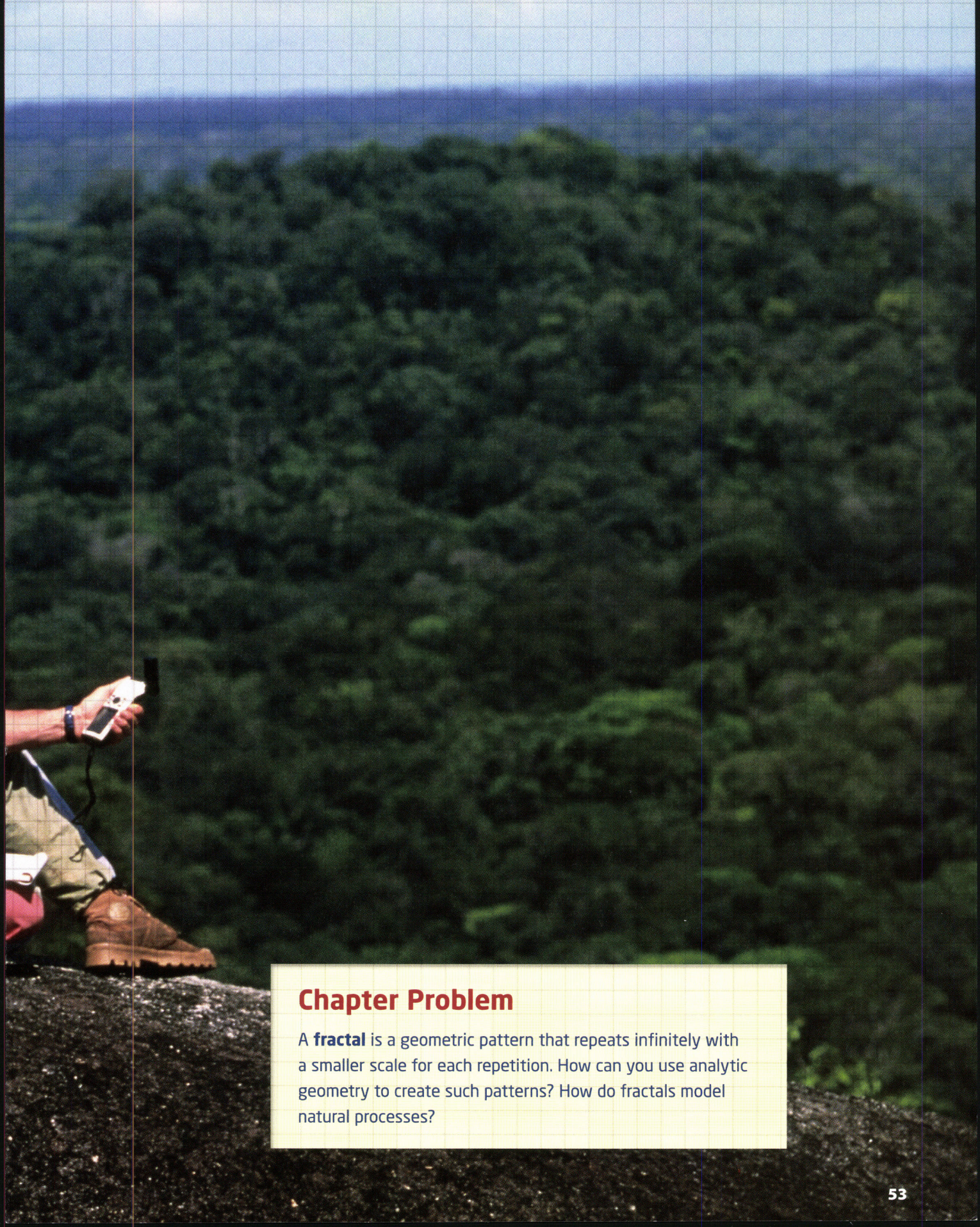
Analytic Geometry

- Develop a formula for the midpoint of a line segment, and use this formula to solve problems.
- Develop a formula for the length of a line segment, and use this formula to solve problems.
- Develop an equation for a circle with centre (0, 0) and radius r.
- Determine the radius of a circle with centre (0, 0) given its equation, write an equation for a circle with centre (0, 0) given its radius, and sketch the circle given its equation in the form $x^2 + y^2 = r^2$.
- Solve problems involving the slope, length, and midpoint of a line segment.

Vocabulary

fractal
Cartesian grid
midpoint
median
equidistant
right bisector
altitude





Chapter Problem

A **fractal** is a geometric pattern that repeats infinitely with a smaller scale for each repetition. How can you use analytic geometry to create such patterns? How do fractals model natural processes?

Get Ready

Solving Equations

You can use opposite operations to isolate a variable in an equation.

1. Solve each equation.

a) $3x + 2 = 14$

b) $7y - 5 = 2y + 10$

c) $\frac{1}{4}z = \frac{1}{3}z - 2$

d) $\sqrt{t} = 0.5$

2. Convert each equation to the form $y = mx + b$.

a) $x - y + 2 = 0$

b) $3x + y - 5 = 0$

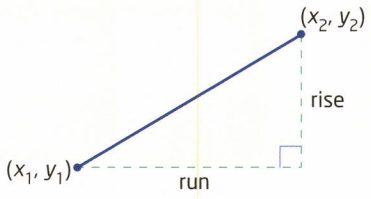
c) $2x - 4y + 7 = 0$

d) $\frac{1}{2}x - 3y + 5 = 0$

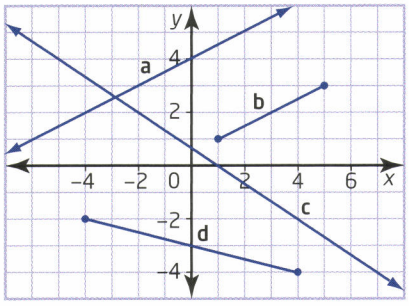
Slope of a Line

You can determine the slope, m, of a line from the coordinates of any two points, (x_1, y_1) and (x_2, y_2), on the line:

$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{y_2 - y_1}{x_2 - x_1}$$



3. Find the slope of each line or line segment.



4. Find the slope of the line through each pair of points.

a) (4, 6) and (12, 10)

b) (−4, 6) and (12, 2)

c) (−5, −4) and (3, −8)

d) (2.5, 6.4) and (9.8, 7.6)

Equation for a Line

- If you know the slope and y-intercept of a line, substitute these values directly into $y = mx + b$ to get an equation for the line.
- If you know the slope and the coordinates of a point on the line, substitute into $y = mx + b$ and solve for the y-intercept, b.
- If you know the coordinates of two points on the line, use these coordinates to calculate the slope. Then, use the slope and the coordinates of either point to solve for the y-intercept, b.

5. Find an equation for the line that

a) has slope -2 and y-intercept 4

b) has slope $\frac{2}{7}$ and y-intercept -14

c) has slope 4 and passes through (6, 3)

d) has slope $-\frac{1}{2}$ and passes through $(-2, 4)$

6. Find an equation for the line that passes through each pair of points.

a) A(1, 1) and B(5, 9)

b) C(−1, 1) and D(−3, −2)

c) E(−4, 1) and F(2, 4)

d) G(5, −8) and H(−1, 4)

Parallel and Perpendicular Lines

- Parallel lines have equal slopes: $m_1 = m_2$.
- The slopes of perpendicular lines are negative reciprocals of each other: $m_2 = -\frac{1}{m_1}$.

7. Find the slope of a line with each property.

a) parallel to the line defined by $y = 3x + 16$

b) parallel to the line defined by $y = -\frac{1}{6}x + 5$

c) perpendicular to the line defined by $y = -4x - 7$

d) perpendicular to the line defined by $y = \frac{3}{4}x + 8$

8. Find an equation for the line that

a) is parallel to the line defined by $y = -3x + 1$ and passes through A(−3, 5)

b) is perpendicular to the line defined by $y = -\frac{3}{2}x - \frac{1}{2}$ and passes through B(2, 3)

c) is parallel to the line defined by $y = -\frac{3}{4}x - \frac{1}{2}$ and passes through C(−5, 1)

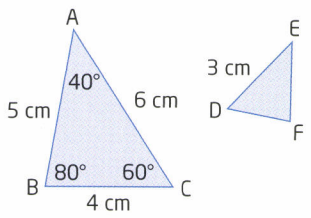
Similar and Congruent Triangles

- Similar geometric figures have the same shape but may differ in size.
- Congruent geometric figures have exactly the same shape and size.

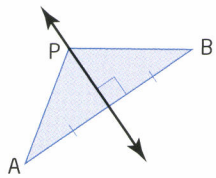
9. △ABC is similar to △EFD.

a) Find the measure of ∠D.

b) Find the length of side EF.



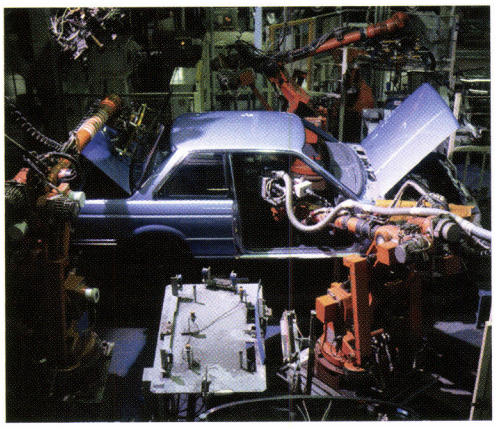
10. Use congruent triangles to show that any point on the **right bisector** of a line segment is the same distance from both endpoints.



2.1 Midpoint of a Line Segment

Cartesian grid
- grid with perpendicular axes

Coordinates on a **Cartesian grid** are a simple and convenient way to specify a location. Such grids have many uses beyond graphs and maps. Machinists use coordinates to direct computer numerical control (CNC) machine tools that shape, drill, and weld parts. Similarly, coordinate systems are essential for programming industrial robots on automated assembly lines.



When you know the coordinates of a set of points, you can calculate the coordinates of related points, such as the **midpoint** between a pair of points.

midpoint
- point that divides a line segment into two equal line segments

Investigate

How can you determine the coordinates of a midpoint?

Tools
- grid paper

Method 1: Use Pencil and Paper

1. Use a Cartesian grid to plot the line segment defined by each pair of endpoints. Label the endpoints with their coordinates. What property do the line segments have in common?
   a) A(−4, 2) and B(6, 2) **b)** C(−3, 0) and D(2, 0)
   c) E(5, −2) and F(−4, −2)

2. Count squares or use a ruler to determine the coordinates of the midpoint of each segment. Label each midpoint with its coordinates. How are the coordinates of the midpoint of each line segment related to the coordinates of its endpoints?

3. Plot and label the line segment defined by each pair of endpoints. What property do the line segments have in common?
   a) G(−4, 2) and H(−4, −6) **b)** J(−1, 7) and K(−1, −2)
   c) L(5, −4) and N(5, 7)

4. Determine the coordinates of the midpoint of each line segment in step 3. Label each midpoint with its coordinates. How are the coordinates of the midpoint of each segment related to the coordinates of its endpoints?

5. Plot and label the line segment defined by each pair of endpoints.
   a) P(1, 1) and Q(7, 5)
   b) R(−5, −4) and S(−1, 0)
   c) T(−3, −4) and U(6, 1)
   d) V(−4, 6) and W(3, 4)

6. Determine the coordinates of the midpoint of each line segment in step 5. Describe how you calculated these coordinates.

7. **Reflect** How are the coordinates of the midpoint of a line segment related to the coordinates of the endpoints? Write an expression for the coordinates of the midpoint of a line segment that has endpoints at (x_1, y_1) and (x_2, y_2).

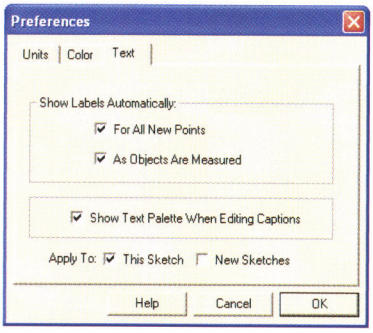
Method 2: Use *The Geometer's Sketchpad®*

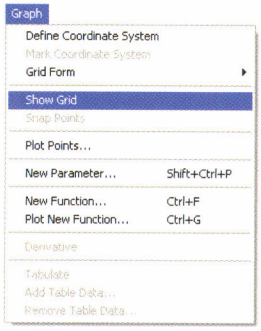
Tools
- computer with *The Geometer's Sketchpad®*

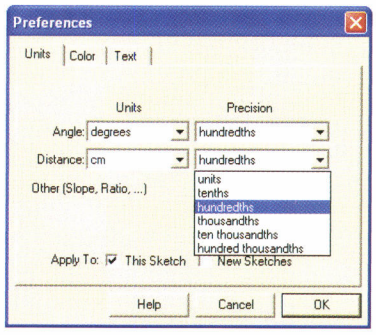
1. Turn on automatic labelling of points. From the **Edit** menu, choose **Preferences**. Click on the **Text** tab. Ensure that **For All New Points** is checked.

   To display a Cartesian grid, open the **Graph** menu and choose **Show Grid**.

   From the **Edit** menu, choose **Preferences;** for distance, choose **cm** units and **hundredths** precision.

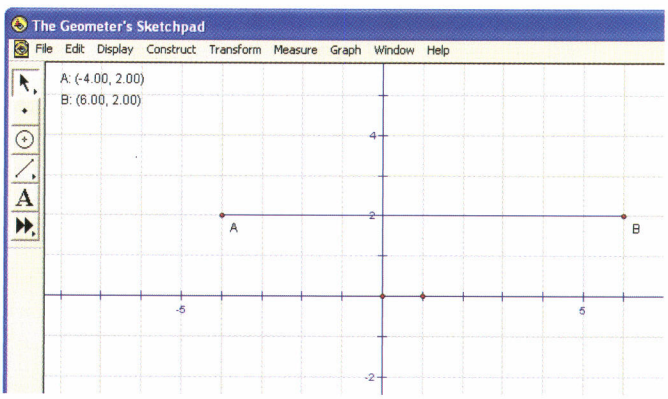






2. From the **Graph** menu, choose **Plot Points**. Enter the coordinates for the point A(−4, 2). Use the tab key to move from the x-coordinate to the y-coordinate. Then, plot the point B(6, 2). To display the coordinates of points A and B, select the points and choose **Coordinates** from the **Measure** menu.

3. Construct the line segment AB by selecting points A and B and then choosing **Segment** from the **Construct** menu. Predict the coordinates of the midpoint of this line segment.



4. Select line segment AB and choose **Midpoint** from the **Construct** menu. Then, select the midpoint and choose **Coordinates** from the **Measure** menu. Check whether the coordinates of the midpoint match your prediction.

5. Construct a line segment with endpoints D(−3, 0) and E(2, 0). Predict the coordinates of the midpoint of DE. Then, use *The Geometer's Sketchpad®* to determine these coordinates. Predict and then determine the coordinates of the midpoint of a line segment with endpoints G(5, −2) and H(−4, −2).

6. **Reflect** What property do line segments AB, DE, and GH have in common? How are the coordinates of their midpoints related to the coordinates of their endpoints?

7. Start a new sketch. Predict and then determine the coordinates of the midpoint of the line segment defined by each pair of endpoints.

   a) A(−4, 2) and B(−4, −6) **b)** D(−1, 7) and E(−1, −2)

   c) G(5, −4) and H(5, 7)

8. **Reflect** What property do line segments AB, DE, and GH have in common? How are the coordinates of their midpoints related to the coordinates of their endpoints?

9. Start a new sketch. Predict and then determine the coordinates of the midpoint of the line segment defined by each pair of endpoints.

   a) A(1, 1) and B(7, 5) **b)** D(−5, −4) and E(−1, 0)

   c) G(−3,−4) and H(6, 1) **d)** J(−4, 6) and K(3, 4)

10. **Reflect** How are the coordinates of the midpoint of a line segment related to the coordinates of the endpoints? Write an expression for the coordinates of the midpoint of a line segment that has endpoints at (x_1, y_1) and (x_2, y_2).

Tools

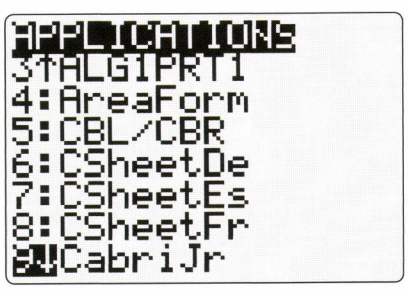
- TI-83 Plus or TI-84 Plus graphing calculator

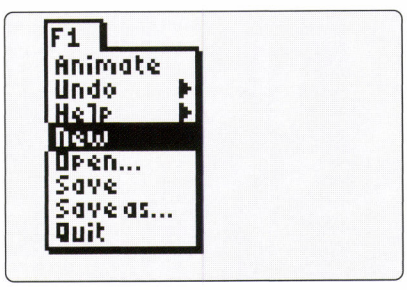
Technology Tip

The position where **CabriJr** appears on the **APPS** screen depends on what other applications have been installed.

Method 3: Use a Graphing Calculator

1. Press APPS, and choose **CabriJr**. Press ENTER when the title screen appears. If you need to clear the screen, press Y= to display the **F1** menu, and choose **New**.

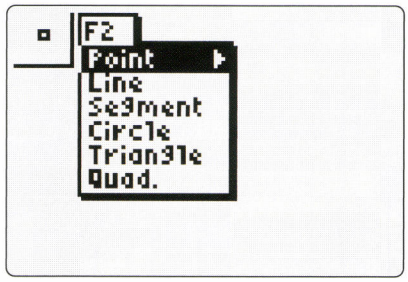




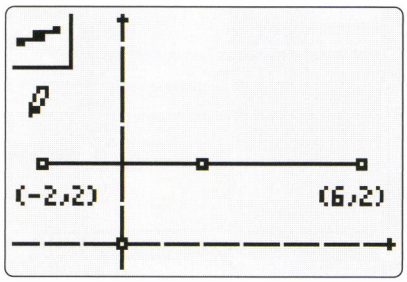
2. If the axes do not appear on the screen, press GRAPH to display the **F5** menu. Highlight **Hide/Show**, press ▸, and choose **Axes**. Press ENTER and then CLEAR. Position the axes so that you can graph the points (−2, 2) and (6, 2). Move the cursor close to the axes. When the axes start flashing, press ALPHA. Then, use the cursor keys to move the axes to the desired position and press ALPHA again.

WWW Go to www.mcgrawhill.ca/links/principles10 and follow the links to install or upgrade Cabri® Jr.

3. Press WINDOW to display the **F2** menu. Move the cursor up to **Point** if it is not already highlighted and press ENTER. Press GRAPH to display the **F5** menu, choose **Coord.&Eq.**, and press ENTER twice to show the coordinates of the plotted point. You can use the cursor keys to move the coordinates label. Press ENTER and then CLEAR. If the coordinates are not exactly (−2, 2), move the cursor to the point. When the point flashes, press ALPHA. Then, use the cursor keys to reposition the point and press ALPHA again.

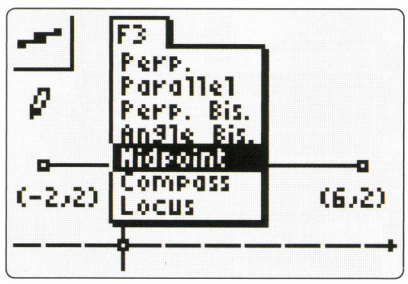


4. Plot and label a point at (6, 2). Press WINDOW to display the **F2** menu. Choose **Segment**, move the cursor to one of the points, and press ENTER. Move the cursor to the other point, press ENTER again, and press CLEAR. Predict the coordinates of the midpoint of the line segment joining the two points.



Technology Tip
Small gaps mark units along the axes.

5. Press ZOOM to display the **F3** menu. Choose **Midpoint** and press ENTER. Move the cursor to the line segment. When the line segment starts flashing, press ENTER again. Now, move the cursor to the midpoint of the segment. Press GRAPH to display the **F5** menu. Choose **Coord.&Eq.**, press ENTER twice, and press CLEAR. Check whether the coordinates of the midpoint match your prediction.



6. Construct a line segment with endpoints (−3, 0) and (2, 0). Predict the coordinates of the midpoint of this line segment. Then, use Cabri® Jr. to determine these coordinates. Predict and then determine the coordinates of the midpoint of a line segment with endpoints (5, −2) and (−2, −2).

Technology Tip

You can also use the cursor keys to switch between options in a dialogue box.

7. **Reflect** What property do the three line segments on the screen have in common? How are the coordinates of their midpoints related to the coordinates of their endpoints?

8. To clear the screen, press Y= and choose **New**. Press 2nd to highlight **NO**, and then press ENTER.

9. Predict and then determine the coordinates of the midpoint of the line segment defined by each pair of endpoints.
   a) $(-4, 4)$ and $(-4, -2)$ **b)** $(-1, 3)$ and $(-1, -1)$
   c) $(5, -2)$ and $(5, 3)$

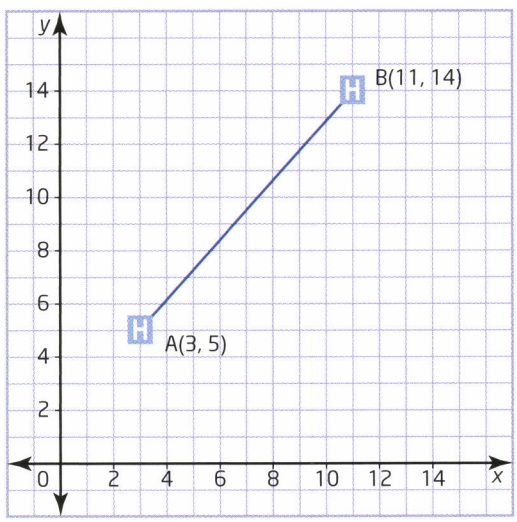
10. **Reflect** What property do the three line segments in step 9 have in common? How are the coordinates of their midpoints related to the coordinates of their endpoints?

11. Start a new graph. Predict and then determine the coordinates of the midpoint of the line segment defined by each pair of endpoints.
   a) $(-2, 2)$ and $(4, 4)$ **b)** $(1, 0)$ and $(5, 3)$
   c) $(-2, -1)$ and $(4, -2)$

12. **Reflect** How are the coordinates of the midpoint of a line segment related to the coordinates of the endpoints? Write an expression for the coordinates of the midpoint of a line segment that has endpoints at (x_1, y_1) and (x_2, y_2).

Example 1 Find a Midpoint

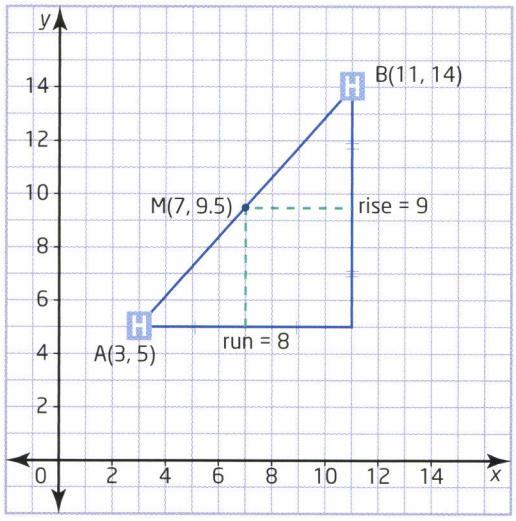
A city has two hospitals, shown on the city map at coordinates A(3, 5) and B(11, 14). The city wants to build a new ambulance station halfway between the two hospitals. Determine the coordinates of this location.



Solution

Method 1: Calculate the Rise and Run

The location of the new ambulance station is the midpoint of the line segment AB. The run between point A and the midpoint is half the run of AB. Similarly, the rise between point A and the midpoint is half the rise of AB.



$$\begin{aligned} \text{run} &= x_2 - x_1 \\ &= 11 - 3 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \text{rise} &= y_2 - y_1 \\ &= 14 - 5 \\ &= 9 \end{aligned}$$

I could use a similar method to divide a line segment into three or more equal parts.

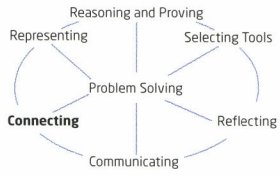
Therefore, the coordinates of the midpoint are

$$\begin{aligned} (x, y) &= \left(x_1 + \frac{\text{run}}{2}, y_1 + \frac{\text{rise}}{2}\right) \\ &= \left(3 + \frac{8}{2}, 5 + \frac{9}{2}\right) \\ &= (3 + 4, 5 + 4.5) \\ &= (7, 9.5) \end{aligned}$$

I can also subtract half the run and half the rise from the coordinates of the second point.

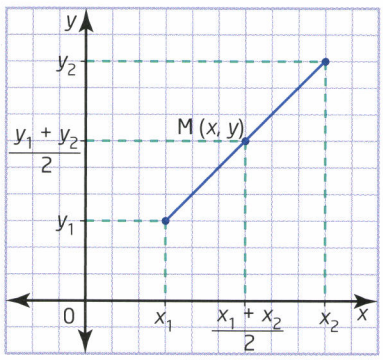
Method 2: Use a Formula

The x-coordinate of the midpoint is equal to the x-coordinate of point A plus half the difference between the x-coordinate of point B and the x-coordinate of point A. So, the x-coordinate of the midpoint is the mean of the x-coordinates of the endpoints of AB. Similarly, the y-coordinate of the midpoint is the mean of the y-coordinates of the endpoints.



Therefore, the coordinates of the midpoint are

$$\begin{aligned} (x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{3 + 11}{2}, \frac{5 + 14}{2}\right) \\ &= \left(\frac{14}{2}, \frac{19}{2}\right) \\ &= (7, 9.5) \end{aligned}$$



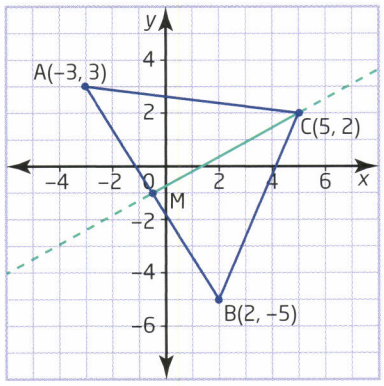
The coordinates of the new ambulance station are (7, 9.5).

median

- line segment joining a vertex of a triangle to the midpoint of the opposite side

Example 2 Median of a Triangle

Determine an equation for the **median** from vertex C for the triangle with vertices C(5, 2), A(−3, 3), and B(2, −5).



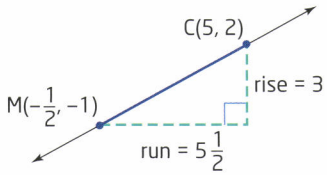
Solution

The median from C goes to the midpoint, M, of the opposite side, AB.

Use the formula from Example 1 to determine the coordinates of this midpoint.

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{-3 + 2}{2}, \frac{3 + (-5)}{2}\right)$$
$$= \left(\frac{-1}{2}, \frac{-2}{2}\right)$$
$$= \left(-\frac{1}{2}, -1\right)$$

Now, find the slope of CM.



$$\text{Slope, } m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2 - (-1)}{5 - \left(-\frac{1}{2}\right)}$$
$$= \frac{3}{\frac{11}{2}}$$
$$= 3 \times \frac{2}{11}$$
$$= \frac{6}{11}$$

If I extend the median on the graph, I can verify that its slope is $\frac{6}{11}$ by checking that the rise is 6 over a run of 11.

Since the point C(5, 2) is on the median, $y = 2$ when $x = 5$. Use these coordinates and the slope to solve for the y-intercept, b.

$$y = mx + b$$
$$2 = \frac{6}{11}(5) + b$$
$$2 = \frac{30}{11} + b$$
$$\frac{22}{11} = \frac{30}{11} + b$$
$$\frac{22}{11} - \frac{30}{11} = b$$
$$\frac{-8}{11} = b$$

I could use the coordinates of the midpoint, M, but the calculation with point A is easier since it has integer coordinates. I could use the coordinates of M to verify my answer.

The median intersects the y-axis at $y = -\frac{8}{11}$.

The y-intercept of the median is $-\frac{8}{11}$.

Therefore, an equation for the median from vertex C is $y = \frac{6}{11}x - \frac{8}{11}$.

Example 3 Equation of a Right Bisector

Two schools are located at the points P(−1, 4) and Q(7, −2) on a town map. The school board is planning a new sports complex to be used by both schools. The board wants to find a location **equidistant** from the two schools. Use an equation to represent the possible locations for the sports complex.

equidistant

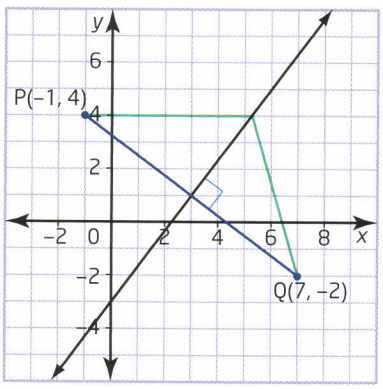
- equally distant

Solution

From the diagram, you can see that a point can be the same distance from both schools without being directly between them. In fact, any point on the **right bisector** of a line segment is equidistant from the endpoints of the segment. The possible locations for the athletic complex lie on the right bisector of PQ.

right bisector

- the line that passes through the midpoint of a line segment and intersects it at a 90° angle



Literacy Connections

A *right bisector* is sometimes called a *perpendicular bisector.*

To determine an equation for the right bisector, find the slope of the bisector and the coordinates of the midpoint of PQ. First, determine the slope of PQ.

$$\begin{aligned} m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - (-2)}{-1 - 7} \\ &= \frac{6}{-8} \\ &= -\frac{3}{4} \end{aligned}$$

Perpendicular lines have slopes that are the negative reciprocals of each other. So, the slope of any line perpendicular to PQ is

$$m_{\perp} = \frac{4}{3}$$

To find the negative reciprocal of a fraction, invert the fraction and use the opposite sign.

The right bisector passes through the midpoint of PQ. Use the midpoint formula to find the coordinates of the midpoint.

$$\begin{aligned} (x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{-1 + 7}{2}, \frac{4 + (-2)}{2}\right) \\ &= \left(\frac{6}{2}, \frac{2}{2}\right) \\ &= (3, 1) \end{aligned}$$

Now, use the coordinates of the midpoint with the slope to solve for the y-intercept of the right bisector.

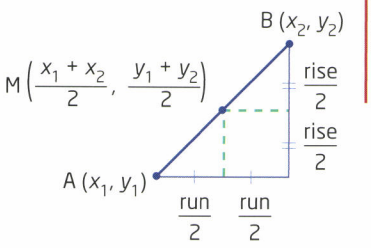
$$\begin{aligned} y &= mx + b \\ 1 &= \frac{4}{3}(3) + b \\ 1 &= 4 + b \\ 1 - 4 &= b \\ -3 &= b \end{aligned}$$

I can use the graph to check that this value for the y-intercept is reasonable.

An equation for the right bisector of PQ is $y = \frac{4}{3}x - 3$. This equation represents the possible locations for the sports complex.

Key Concepts

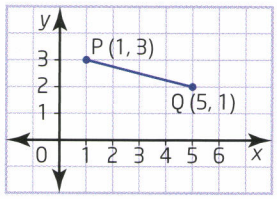
- The midpoint of a line segment can be found by adding half of the run and half of the rise to the coordinates of the first endpoint.
- Each coordinate of the midpoint of a line segment is the mean of the corresponding coordinates of the endpoints.



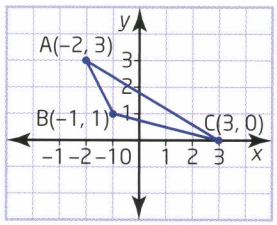
- The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- To find an equation for the median of a triangle, first find the coordinates of the midpoint of the side opposite the vertex. Use the coordinates of the midpoint and the vertex to calculate the slope of the median. Then, substitute the slope and the coordinates of either point into $y = mx + b$ to solve for the median's y-intercept.
- To find an equation for the right bisector of a line segment, first find the slope and midpoint of the segment. Use the line segment's slope to calculate the slope of a perpendicular line. Then, substitute this slope and the coordinates of the midpoint into $y = mx + b$ to solve for the right bisector's y-intercept.

Communicate Your Understanding

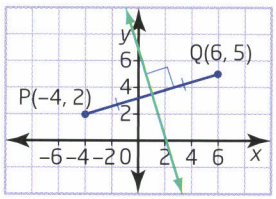
C1 Describe two methods for finding the midpoint of this line segment.



C2 Describe how to determine an equation for the median from vertex A of △ABC.



C3 Describe how to determine an equation for the right bisector of line segment PQ.

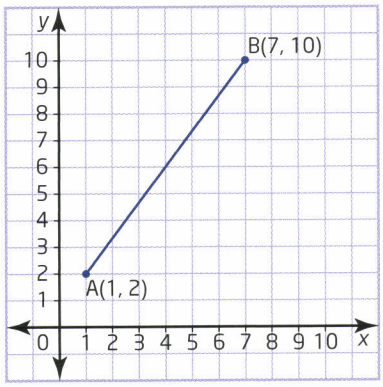


Practise

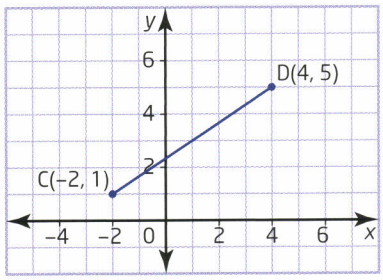
For help with questions 1 to 3, see Example 1.

1. Determine the coordinates of the midpoint of each line segment.

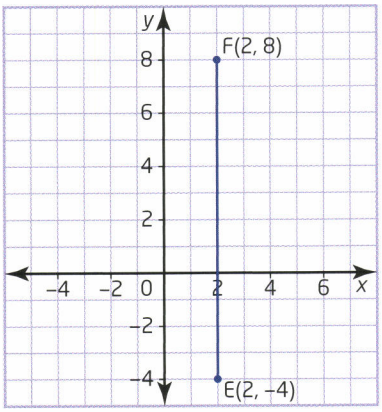
a)



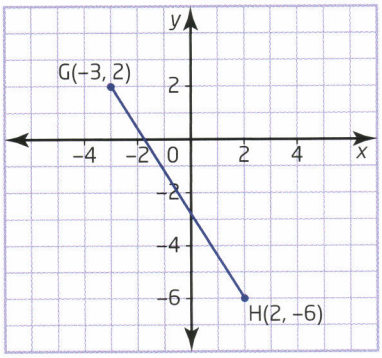
b)



c)



d)



2. Determine the coordinates of the midpoint of the line segment defined by each pair of endpoints.

a) J(5, 7) and K(3, 9)

b) L(−1, 0) and M(1, −6)

c) N(−2, −4) and P(−2, 8)

d) Q(−3, −3) and R(−1, −7)

3. Determine the coordinates of the midpoint of the line segment defined by each pair of endpoints.

a) J(0.2, 1.5) and K(3.6, 0.2)

b) N(−1.4, −3.2) and P(0.6, −5.3)

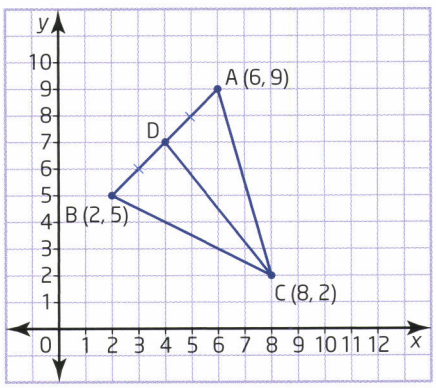
c) $\mathrm{L}\left(\frac{1}{2}, \frac{5}{2}\right)$ and $\mathrm{M}\left(\frac{3}{2}, -\frac{5}{2}\right)$

d) $\mathrm{Q}\left(-\frac{3}{8}, \frac{1}{8}\right)$ and $\mathrm{R}\left(2, -\frac{7}{8}\right)$

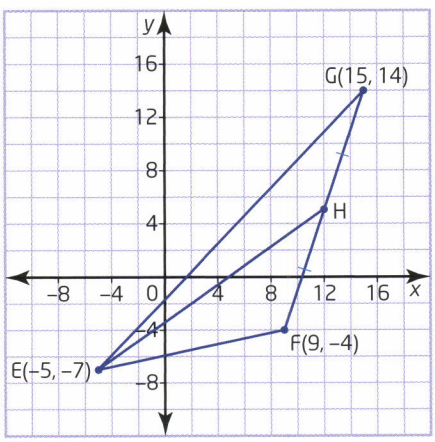
For help with question 4, see Example 2.

4. Find the slope of each median shown.

a)



b)



Connect and Apply

5. A charity is organizing a fundraising run along a straight section of highway. On the grid of a roadmap, the starting point is at (23.6, 38.0) and the finish line is at (79.4, 43.8). The charity wants to set up a checkpoint table with water for the runners at the halfway point. Find the coordinates of this checkpoint.

6. The endpoints of the diameter of a circle are P(−7, −4) and Q(−1, 10). Find the coordinates of the centre of this circle.

7. Use Technology Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 6. Describe the method you used.

8. The vertices of △ABC are A(4, 4), B(−6, 2), and C(2, 0). Find an equation in slope *y*-intercept form for the median from vertex A.

9. Use Technology Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 8. Describe the method you used.

Technology Tip

You can use geometry software to display an equation for a line:

- With *The Geometer's Sketchpad*®, choose **Equation** from the **Measure** menu.
- With Cabri® Jr., choose **Coord.&Eq.** from the **F5** menu.

10. For the triangle with vertices P(−2, 0), Q(4, 6), and R(5, −3), find an equation for the median from

a) vertex P **b)** vertex Q

11. Use Technology Use geometry software to check your answer to question 10. Describe your method.

12. Write an expression for the coordinates of the midpoint of the line segment with endpoints P(a, b) and Q($3a$, $2b$). Explain your reasoning.

13. A line segment with one end at C(6, 5) has midpoint M(4, 2).

a) Determine the coordinates of the other endpoint, D.

b) Explain your solution.

c) Describe a method you could use to check your answer to part a).

14. One endpoint of a diameter of a circle centred on the origin is (−3, 4). Find the coordinates of the other endpoint of this diameter.

15. One radius of a circle has endpoints D(2, 4) and E(−1, 2).

a) Find a possible endpoint for the diameter that contains this radius.

b) Explain why there are two possible answers in part a).

16. Determine an equation for the right bisector of the line segment with endpoints P(−5, −2) and Q(3, 6).

17. A telecommunications company wants to build a relay tower that is the same distance from two adjacent towns. On a local map, the towns have coordinates (2, 6) and (10, 0).

a) Explain how you could use a right bisector to find possible locations for the tower.

b) Find an equation for this bisector.

18. Use Technology Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 17. Describe the method you used.

19. a) Draw △ABC with vertices A(−2, 0), B(8, 8), and C(4, −2).

b) Draw the median from vertex A. Then, find an equation in slope *y*-intercept form for this median.

c) Draw the right bisector of BC. Then, find an equation for this right bisector.

d) Use your drawing to check your answers for parts b) and c).

20. **a)** Draw $\triangle PQR$ with vertices P(0, 0), Q(16, 0), and R(0, 16).

b) Construct the midpoints of PQ, QR, and PR, and label them S, T, and U, respectively.

c) Join the midpoints to form $\triangle STU$. The length of a line segment joining the midpoints of two sides of a triangle is half the length of the third side. Use this property to show that $\triangle STU$ is congruent to all three of the other triangles inside $\triangle PQR$.

d) Compare the area of $\triangle STU$ to the area of $\triangle PQR$.

e) Shade $\triangle STU$. Construct and label the midpoint of each side of the three other triangles inside $\triangle PQR$. Join the midpoints to create a set of even smaller triangles.

f) Compare the area of one of these triangles to the area of $\triangle STU$ and to the area of $\triangle PQR$.

21. **Chapter Problem** Question 20 uses a procedure developed by the Polish mathematician Waclaw Sierpinski in 1915. This procedure produces a fractal known as Sierpinski's triangle or Sierpinski's gasket.

a) Use a library or the Internet to learn more about Sierpinski's triangle.

b) Describe the procedure for producing this fractal. Does the procedure work with any shape of triangle? Explain.

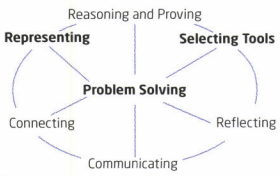
c) Sketch the first four stages in shading a triangle using Sierpinski's method.

d) Explain why Sierpinski's triangle is a fractal.

Go to www.mcgrawhill.ca/links/principles10 and follow the links to learn more about Sierpinski's triangle.

Extend

22. A is the midpoint of BC, D is the midpoint of AC, and E is the midpoint of AD. ED is 2 units in length. What is the length of BC?



23. A line segment has endpoints A(2, 1) and B(11, 19).

a) Find the coordinates of the two points that divide the segment into three equal parts.

b) Describe the method that you used in part a).

24. In $\triangle ABC$, P(0, 2) is the midpoint of side AB, Q(2, 4) is the midpoint of BC, and R(1, 0) is the midpoint of AC.

a) Find the coordinates of A, B, and C. (Hint: Use the properties of a line segment joining the midpoints of two sides of a triangle.)

b) Use the midpoint formula to check the coordinates you calculated in part a).

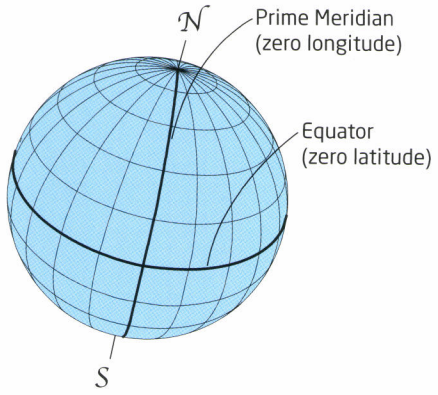
25. In three dimensions, the location of a point can be represented by the ordered triple (x, y, z).

a) Find the coordinates of the midpoint of the line segment with endpoints A(2, 3, 1) and B(6, 7, 5).

b) Write an expression for the coordinates of the midpoint of the line segment with endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2).

26. Suppose that the relay tower in question 17 is to serve three towns instead of two. Describe how you could find a location that is equidistant from all three towns. Can there be more than one such location? Explain, using a diagram to support your answer.

27. Geographers and navigators use a spherical coordinate system with lines of latitude that are parallel to the equator and lines of longitude that are perpendicular to the lines of latitude and meet at Earth's poles.



a) Explain why the formula for the midpoint of a line segment will not always give accurate results with the longitude-latitude coordinate system.

b) Search the Internet to find a Web site that calculates the distance between two points from their latitudes and longitudes. Use the site's calculator to show that the point at the mean of the latitudes and longitudes of two points is not actually equidistant from the two points.

WWW Go to www.mcgrawhill.ca/links/principles10 and follow the links to experiment with distance calculators.

28. Decide whether each statement is always true, sometimes true, or never true. Justify your answers.

a) Two line segments with the same midpoint have the same length.

b) Two parallel line segments have the same midpoint.

c) The midpoint of a line segment is the only point that divides it into two equal parts.

d) A point equidistant from the endpoints of a line segment is the midpoint of the line segment.

29. The endpoints of line segment PQ are P(3, –4) and Q(11, c). The midpoint of PQ is M(d, 3). Find the values of c and d.

30. Math Contest The number of arrangements of five letters from the word *magnetic* that end with a vowel is

A 360 **B** 420 **C** 840 **D** 2520 **E** 7560

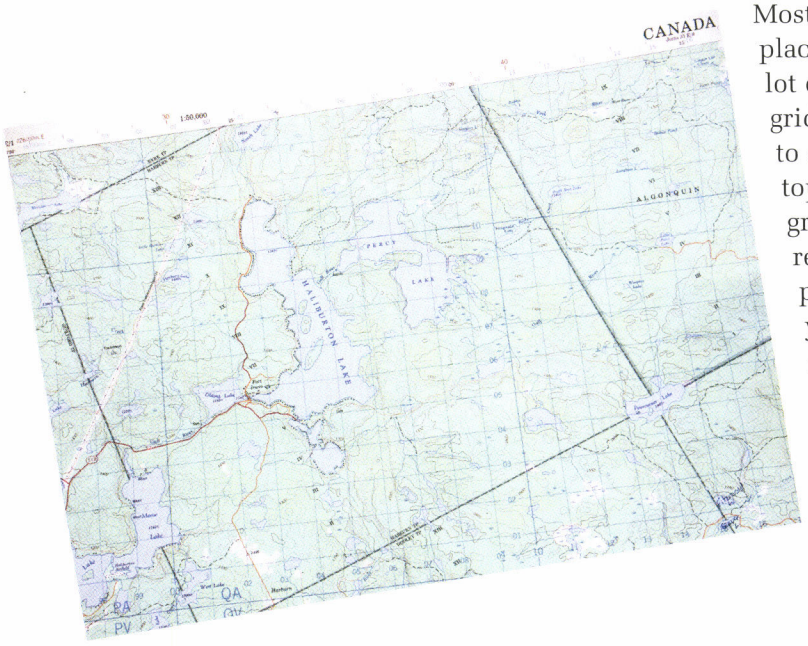
31. Math Contest If $2^5 + 2^5 = 2^x$, the value of x is

A 3 **B** 5 **C** 6 **D** 8 **E** 10

Making Connections

The branch of mathematics known as analytic geometry started when French mathematician René Descartes (1596–1650) invented a system of numerical coordinates for describing locations on a rectangular grid. The Cartesian grid let mathematicians use algebra to analyse geometric shapes. Before Descartes, all geometric properties were proved using logical reasoning based on the work of the Greek mathematician Euclid (around 300 B.C.E.) and his followers. Analytic geometry is the basis of applications such as computer-aided design (CAD) and computer-generated imaging (CGI). These applications are widely used for engineering, architecture, medical tests, cartoon animation, and special effects in movies.

2.2 Length of a Line Segment

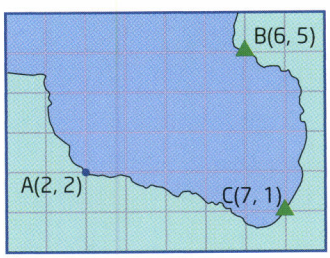


Most maps have a grid for locating places on the map. Maps that show a lot of detail usually have Cartesian grids with a scale that make it easy to estimate distances. For example, topographic maps commonly have grids where each side of a square represents 1 km. If you are planning a hiking or canoe trip, you can easily count squares to get a rough estimate of the length of possible routes. You can also use coordinates to calculate the exact distance between two points.

Investigate

How can you use coordinates to calculate distances?

Jan and Tara are planning a canoe trip in Georgian Bay. The dock where they will launch the canoe is at point A(2, 2) on their map. The campsite where they will stay the first night is at B(6, 5), and the campsite for the second night is at C(7, 1).



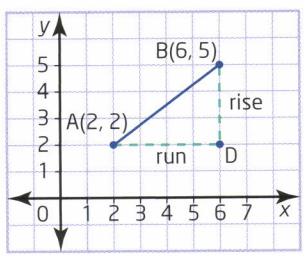
Tools

- grid paper

Method 1: Use Pencil and Paper

1. Plot points A and B on grid paper. Draw a line segment joining points A and B. What does this line segment represent?

2. Draw a right triangle below line segment AB to show its run and rise. How are the run and rise of AB related to the coordinates of its endpoints?



3. Find the coordinates of the vertex of the right angle in the run-rise triangle. Label this vertex D. How are the coordinates of D related to the coordinates of points A and B?

4. Find the lengths of sides AD and BD. Then, use the Pythagorean theorem to determine the length of AB.

5. Each side of a grid square represents an actual distance of 4.0 km. How far will Jan and Tara have to paddle on the first day of their trip?

6. **Reflect** How is the length of line segment AB related to its run and rise? How is this length related to the coordinates of points A and B?

7. Calculate the distance from the campsite at B to the campsite at C. Do you have to change your method for calculating the length if the run or rise is negative? Explain.

8. Calculate the distance from the campsite at C back to the dock at A.

9. **Reflect** How is the length, d, of the line segment with endpoints (x_1, y_1) and (x_2, y_2) related to the run and rise of the segment? Write an equation showing how d is related to the coordinates of the endpoints.

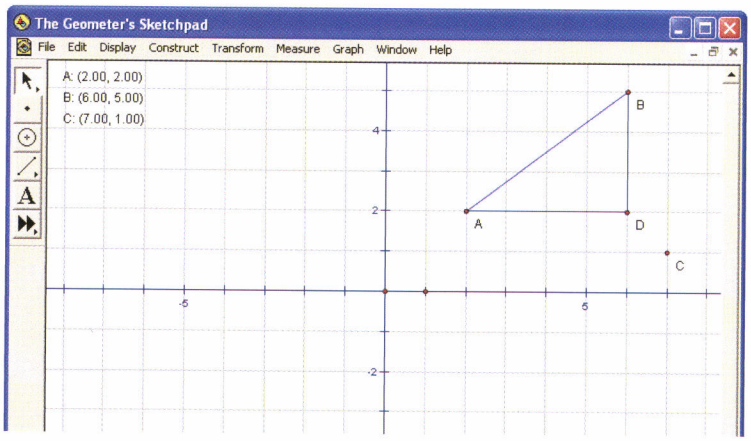
Method 2: Use *The Geometer's Sketchpad*®

Tools
- computer with *The Geometer's Sketchpad*®

1. From the **Graph** menu, choose **Show Grid**. From the **Edit** menu, choose **Preferences**; for distance, choose **cm** units and **hundredths** precision. Check that automatic labelling of points is turned on.

2. From the **Graph** menu, choose **Plot Points**. Then, enter the coordinates for A(2, 2), B(6, 5), and C(7, 1). Display the coordinates by opening the **Measure** menu and choosing **Coordinates**. Construct line segment AB.

3. Construct a right triangle below line segment AB to show its run and rise. How are the run and rise of AB related to the coordinates of its endpoints?



4. Display the coordinates of the vertex, D, of the right angle in the run-rise triangle. How are they related to the coordinates of points A and B?

5. Calculate the lengths of sides AD and BD. Then, use the Pythagorean theorem to determine the length of AB. Check your calculations by selecting the segments and choosing **Length** from the **Measure** menu.

6. Each side of a grid square represents an actual distance of 4.0 km. How far will Jan and Tara have to paddle on the first day of their trip?

7. **Reflect** How is the length of line segment AB related to its run and rise? How is this length related to the coordinates of points A and B?

8. Use the Pythagorean theorem to calculate the distance from the campsite at B to the campsite at C. Use the **Measure** menu to check your answer. Do you have to change your method for calculating the length if the run or rise is negative? Explain.

9. Calculate the distance from the campsite at C back to the dock at A. Use the **Measure** menu to check your answer.

10. **Reflect** How is the length, d, of the line segment with endpoints (x_1, y_1) and (x_2, y_2) related to the run and rise of the segment? Write an equation showing how d is related to the coordinates of the endpoints.

Tools

- TI-83 Plus or TI-84 Plus graphing calculator

Method 3: Use a Graphing Calculator

1. Press APPS and choose **CabriJr**. Press ENTER when the title screen appears. If you need to clear the screen, press Y= to display the **F1** menu and choose **New**.

2. Press GRAPH to display the **F5** menu. Choose **Hide/Show**, press ▶, and choose **Axes**. Press ENTER and then CLEAR. Move the cursor close to the axes. When the axes start flashing, press ALPHA. Then, use the cursor keys to move the axes to the lower left corner of the screen. Press ALPHA again.

3. Press WINDOW to display the **F2** menu. Choose **Point**. Use the cursor keys to move the pen cursor to (2, 2) and press ENTER. Press GRAPH to display the **F5** menu, choose **Coord.&Eq.**, and press ENTER twice to show the coordinates of the plotted point. You can use the cursor keys to move the coordinates label. Press ENTER and then CLEAR. If the coordinates are not exactly (2, 2), move the cursor to the point A. When the point flashes, press ALPHA. Then, use the cursor keys to reposition the point and press ALPHA again.

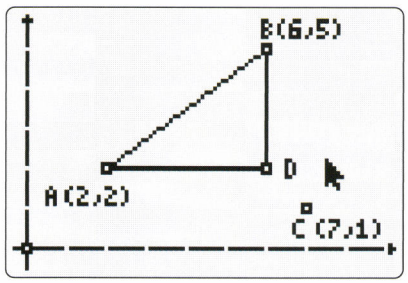
4. To add a letter label to a point, choose **Alph-Num** from the **F5** menu. Move the cursor near the point. Press ENTER and then ALPHA followed by the key for the letter. Then, press ENTER to lock the label in place. To move the label, press CLEAR and move the cursor toward the letter until it starts flashing. Then, press ALPHA, use the cursor keys to move the label to the desired location, and press ENTER.

Technology Tip

When moving a label or geometric shape, you can press either ENTER or ALPHA to set the position.

5. Plot and label points B(6, 5) and C(7,1). Press WINDOW to display the **F2** menu. Choose **Segment**, move the cursor to A(2, 2), and press ENTER. Move the cursor to B(6, 5), press ENTER again, and then press CLEAR.

6. Use the **F2** menu to construct a point D and two line segments forming a right triangle that shows the run and rise of line segment AB. How are the run and rise of segment AB related to the coordinates of its endpoints? How are the coordinates of point D, the vertex of the right angle, related to the coordinates of points A and B?



7. Find the lengths of sides AD and BD. Then, use the Pythagorean theorem to calculate the length of AB. To check this calculation, choose **Measure** from the **F5** menu and then choose **D.&Length**. Move the cursor toward segment AB until it flashes, and press ENTER. Use the cursor keys to position the measurement, if necessary; then, press ENTER again.

8. Each side of a grid square represents an actual distance of 4.0 km. How far will Jan and Tara have to paddle on the first day of their trip?

9. **Reflect** How is the length of line segment AB related to its run and rise? How is this length related to the coordinates of points A and B?

10. Use the Pythagorean theorem to calculate the distance from the campsite at B to the campsite at C. Measure the length as described above to check your answer. Do you have to change your method for calculating the length if the run or rise is negative? Explain.

11. Calculate the distance from the campsite at C back to the dock at A. Measure the length to check your answer.

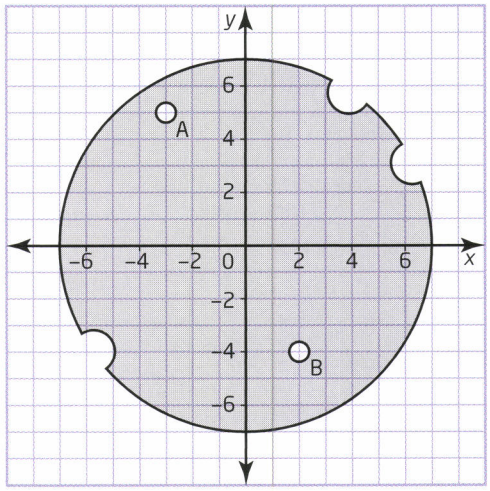
12. **Reflect** How is the length, d, of the line segment with endpoints (x_1, y_1) and (x_2, y_2) related to the run and rise of the segment? Write an equation showing how d is related to the coordinates of the endpoints.

Example 1 Calculate a Length

Literacy Connections

A cam is a rotating mechanical part shaped to guide the motion of another part of a machine. Usually, the controlled part connects to the side of the cam or runs against its edge.

To make round parts, programmable machine tools often use a coordinate system with the origin at the centre of the part. How far apart are the centres of the mounting holes A and B in this cam? The coordinates are in centimetres. Round your answer to the nearest tenth.

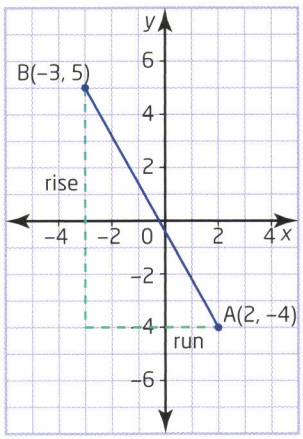


Solution

Applying the Pythagorean theorem gives $AB^2 = \text{run}^2 + \text{rise}^2$. The run of AB is equal to the difference between the x-coordinate of B and the x-coordinate of A. Similarly, the rise of AB is equal to the difference between the corresponding y-coordinates. Therefore,

$$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-3 - 2)^2 + [5 - (-4)]^2} \\ &= \sqrt{(-5)^2 + 9^2} \\ &= \sqrt{25 + 81} \\ &= \sqrt{106} \\ &\doteq 10.3 \end{aligned}$$

The centres of the mounting holes are about 10.3 cm apart.



Example 2 Compare Distances

An air ambulance service uses a grid system to help estimate flying times and fuel requirements. Coordinates on this grid are distances in kilometres east and north of a reference point on the lower left corner of a map of northern Ontario. A helicopter ambulance picks up a patient at point P(96, 197). The nearest hospitals that can provide the treatment the patient needs are in Timmins at T(200, 296) and Sudbury at S(232, 80).

a) To which hospital should the helicopter take the patient?

b) List any assumptions you made for your answer.

Solution

a) First, find the distance to each hospital.

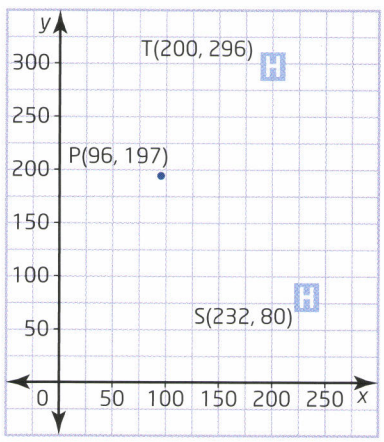
For the Timmins hospital:

$$\begin{aligned} PT &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(200 - 96)^2 + (296 - 197)^2} \\ &= \sqrt{104^2 + 99^2} \\ &= \sqrt{20\,617} \\ &\doteq 144 \end{aligned}$$

For the Sudbury hospital:

$$\begin{aligned} PS &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(232 - 96)^2 + (80 - 197)^2} \\ &= \sqrt{136^2 + (-117)^2} \\ &= \sqrt{32\,185} \\ &\doteq 179 \end{aligned}$$

The helicopter should go to the Timmins hospital because it is closer to the pick-up point.



b) The decision to go to the closer hospital assumes that the helicopter can travel in a straight line to either hospital. The decision also assumes that weather will not affect the flying times or prevent a landing at the closer hospital.

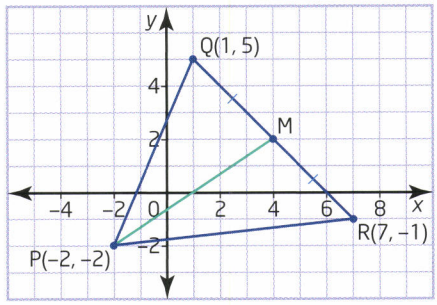
Example 3 Find the Length of a Median

Find the length of the median from P for a triangle with vertices P(−2, −2), Q(7, −1), and R(1, 5).

Solution

The median is the line segment that joins P to the midpoint, M, of QR. To find the coordinates of M, substitute the coordinates of Q and R into the midpoint formula.

$$\begin{aligned}(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\\ &= \left(\frac{7 + 1}{2}, \frac{-1 + 5}{2}\right)\\ &= \left(\frac{8}{2}, \frac{4}{2}\right)\\ &= (4, 2)\end{aligned}$$



Now, substitute the coordinates of P(−2, −2) and M(4, 2) into the length formula.

$$\begin{aligned}PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &= \sqrt{[4 - (-2)]^2 + [2 - (-2)]^2}\\ &= \sqrt{6^2 + 4^2}\\ &= \sqrt{36 + 16}\\ &= \sqrt{52}\end{aligned}$$

The length of the median from vertex P is $\sqrt{52}$.

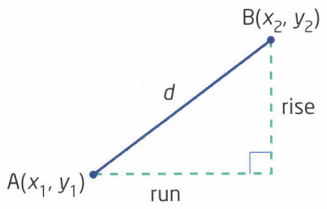
Key Concepts

- You can calculate the length, d, of a line segment using its run and rise:

  $d = \sqrt{(\text{run})^2 + (\text{rise})^2}$

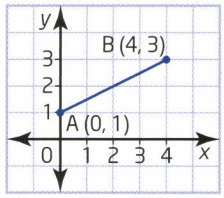
- You can also determine the length of a line segment directly from the coordinates of its endpoints, (x_1, y_1) and (x_2, y_2):

  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Communicate Your Understanding

C1 Describe how you can find the length of the line segment joining the points A(0, 1) and B(4, 3).



C2 When you use the formula for the length of a line segment, does it matter which point is represented by (x_1, y_1) and which point is represented by (x_2, y_2)? Use an example to explain your reasoning.

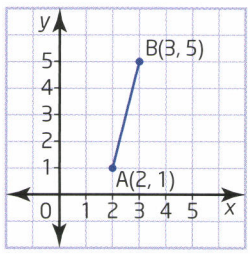
C3 Explain why the expression $(x_2 - x_1)^2 + (y_2 - y_1)^2$ never has a negative value.

Practise

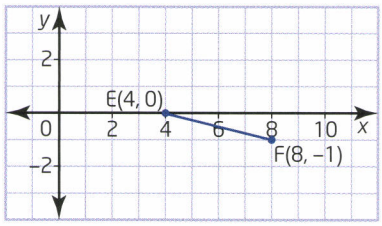
For help with questions 1 to 3, see Examples 1 to 3.

1. Estimate the length of each line segment from its graph. Then, calculate the exact length.

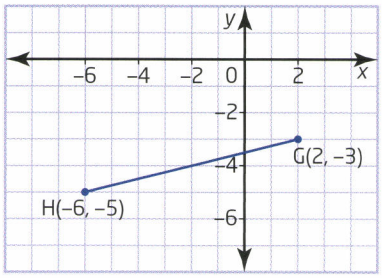
a)



b)



c)



2. Calculate the length of the line segment defined by each pair of endpoints.

a) A(−6, −2) and B(4, 3)

b) C(−2, 0) and D(7, −3)

c) E(−5, −6) and F(−1, −2)

d) G(0, 5) and H(8, −1)

3. Calculate the length of the line segment defined by each pair of endpoints.

a) J(2.1, 8.3) and K(−4.5, −4.7)

b) L(−4.2, −5.1) and M(11.6, 9.2)

c) $N\left(\frac{1}{2}, \frac{5}{2}\right)$ and $P\left(\frac{3}{2}, -\frac{5}{2}\right)$

Connect and Apply

4. On a city map, the coordinates of two department stores are (4, 3) and (1, 7). How far apart are the stores if each unit on the map represents 1 km?

5. On a street map of his town, Jordan's house has coordinates (8, 1). The town's two high schools are at (0, 5) and (6, 11).

a) Which school is closer to Jordan's house?

b) Describe a method you could use to check your answer to part a).

6. The vertices of △ABC are A(2, 5), B(−6, −1), and C(10, −1).

a) Determine the length of each side of this triangle.

b) Determine the perimeter of the triangle.

c) Classify the triangle.

7. a) Show that the triangle with vertices D(−1, 0), E(1, 0), and F(0, $\sqrt{3}$) is equilateral.

b) List the coordinates of the vertices of another equilateral triangle.

8. Determine the length of the median from vertex J in the triangle with vertices J(−2, −2), K(−3, 2), and L(1, 3).

9. **Use Technology** Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 8.

10. Determine the area of the right triangle with vertices R(4, 4), S(−2, −2), and T(10, −2).

11. **Use Technology** Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 10.

12. Use the length formula to verify that C(−5, −1) is the midpoint of the line segment joining A(−2, 5) and B(−8, −7).

13. A line segment has endpoints K(−2, 7) and L(4, −2).

a) Find the coordinates of the midpoint of this line segment.

b) Use the length formula to verify your answer to part a).

14. An architect's drawing shows a pipe running diagonally under a basement floor from a floor drain to a sewer connection. The floor drain is at a point 2 m east and 2 m north of the southwest corner of the basement. The sewer connection is 10 m east and 17 m north of the corner. The pipe costs $3.15 per metre, including taxes. How much should the builder budget for pipe for the floor drain?

15. **a)** Draw a triangle with vertices P(−3, −4), Q(5, 1), and R(2, 7).

b) Determine the coordinates of the midpoints of PQ and PR. Label these midpoints S and T.

c) Show that ST is half the length of QR.

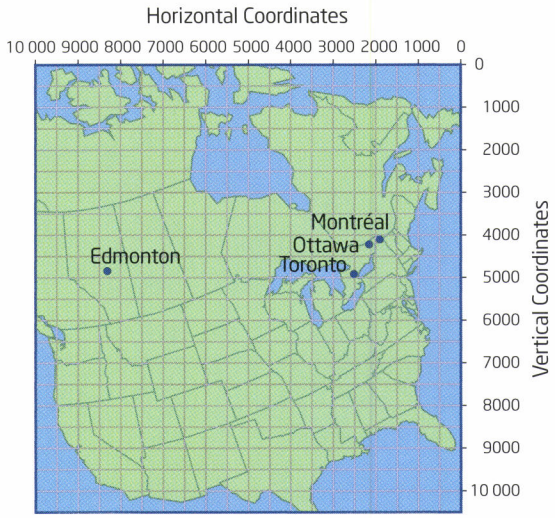
d) Show that ST is parallel to QR.

e) Show that the triangle formed by joining the midpoints of the sides of △PQR is similar to △PQR.

16. **Use Technology** Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 15.

17. The charges for most long-distance telephone calls used to be based on the distance between the two stations and the duration of the call. To determine the distance, telephone companies used a rectangular coordinate grid with its origin located off the northeast coast of Canada. The coordinates in this system indicate the horizontal and vertical distance from the grid's origin. This table lists the telephone coordinates, converted to kilometres, for four cities.

City	Coordinates
Edmonton	(3978, 2520)
Montréal	(1015, 2104)
Ottawa	(1142, 2232)
Toronto	(1268, 2540)



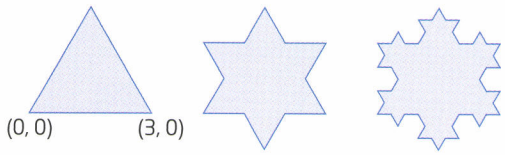
a) Calculate the distance, to the nearest kilometre, between Edmonton and Ottawa, between Montréal and Toronto, and between Edmonton and Toronto.

b) Research the flying distances between these cities. How accurate is the telephone coordinate system?

18. Chapter Problem Here are the first three stages of a Koch snowflake, named after the Swedish mathematician Niels Fabian Helge von Koch (1870–1924). This snowflake starts with an equilateral triangle. At each stage, the middle third of each side is replaced by two segments, each equal in length to the segment they replace.

a) Find the coordinates of the three new vertices on the bottom of the snowflake in the second stage.

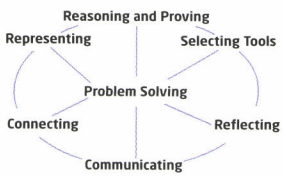
b) Are Koch snowflakes fractals? Explain.



Go to www.mcgrawhill.ca/links/principles10 and follow the links to learn more about Koch snowflakes.

Achievement Check

19. Lightning starts a forest fire at a point with map coordinates F(23, 25). The nearest towns have coordinates A(15, 19) and B(23, 17).



a) Which town is at greater risk from the fire?

b) Describe any assumptions you made for your answer to part a).

c) At a campground midway between the two towns, a camper learns about the fire from a newscast. The camper estimates that he is about 9 km from the fire. Is the camper correct? Justify your response.

Extend

20. The point A(x, 1) is 5 units from the point (2, 6).

a) Find a possible value for x.

b) Is this value the only solution? Explain.

21. a) On a grid, draw line segments with the lengths listed below. Describe your method. How do you know that each line segment is the correct length?

i) $\sqrt{2}$ **ii)** $\sqrt{5}$ **iii)** $\sqrt{13}$ **iv)** $\sqrt{41}$

b) Compare the line segments you drew to those drawn by a classmate. Verify that both sets of line segments have the correct lengths.

22. List four points that satisfy each condition and have integer coordinates.

a) 5 units from the origin

b) 5 units from the point (2, 1)

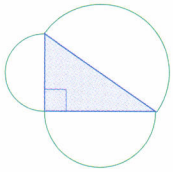
c) 10 units from the point (−5, −2)

23. Sally has hidden her brother's birthday present somewhere in the backyard. When writing instructions for finding the present, she used a coordinate system with each unit on the grid representing 1 m. The positive y-axis of this grid points north. The instructions read "Start at the origin, walk halfway to (8, 6), turn 90° left, and then walk twice as far." Where is the present? How far is it from the origin?

24. Math Contest The number of arrangements of five letters from the word *magnetic* that contain the word *net* is

A 60 **B** 100 **C** 360 **D** 630 **E** 720

25. Math Contest Show that the area of the larger semicircle is equal to the sum of the areas of the two smaller semicircles.



2.3 Apply Slope, Midpoint, and Length Formulas

Ideally, the route of a power line should be as short as possible. A shorter route reduces the construction cost as well as the energy losses due to the resistance of the wire. Engineers use analytic geometry to find the best route for the transmission lines that deliver electricity throughout the province. Analytic geometry is also a powerful tool for designing roads, buildings, pipelines, industrial machinery, and consumer products.

This section shows how to apply geometry and algebra to a variety of problems. Many of these problems involve several steps that require different skills. Developing a problem solving process is particularly important for dealing with such problems. These four steps can help you:

1. Understand the problem.
2. Choose a strategy.
3. Carry out the strategy.
4. Reflect.

Investigate

How can you construct the shortest line segment from a point to a line?

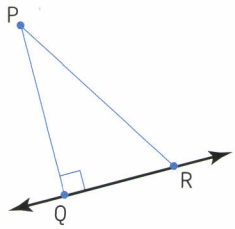
Tools

- grid paper
- protractor or compasses

Method 1: Use Pencil and Paper

1. Draw a line and any point P not on the line.
2. Describe how you could draw a line segment from the point to the line so that the line segment is as short as possible.

3. Draw a line segment from P to meet the line at a right angle. Label the vertex of the right angle Q. Choose any other point on the line and label it R.



4. How are the lengths of PQ, QR, and PR related? Explain why the line segment PQ must be shorter than a line segment joining P to any other point on the line.

5. **Reflect** What property does the shortest line segment from a point to a line have?

Method 2: Use *The Geometer's Sketchpad*®

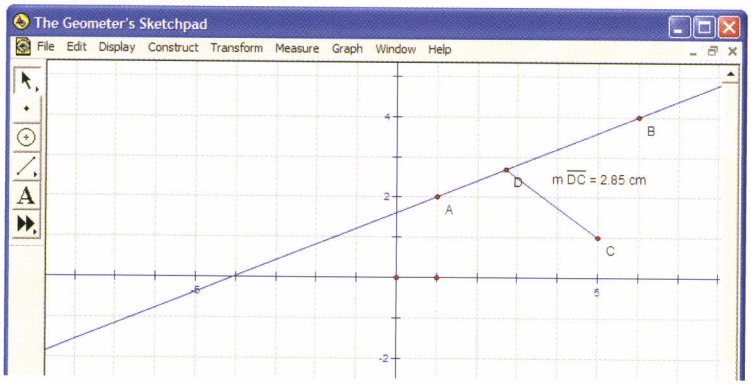
Tools
- computer with *The Geometer's Sketchpad*®

1. Plot two points, A and B, and construct the line through them. Plot a point C that is not on the line.

2. Describe how you could draw a line segment from point C to the line so that the segment is as short as possible.

3. Construct a point D anywhere on the line through points A and B. Then, construct the line segment CD.

4. With segment CD selected, choose **Length** from the **Measure** menu.



5. Slide the point D along the line until the distance is as small as possible. Estimate the measure of ∠ADC.

6. Measure ∠ADC. Was your estimate accurate?

7. **Reflect** What property does the shortest line segment from a point to a line have?

Tools

- TI-83 Plus or TI-84 Plus graphing calculator

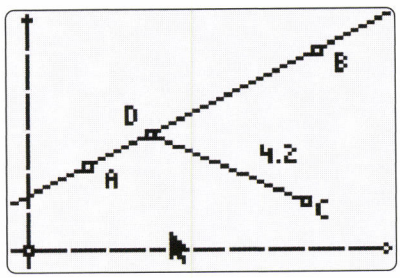
Method 3: Use a Graphing Calculator

1. Start the Cabri® Jr. application. Clear the screen, if necessary.

2. To construct a line, choose **Line** from the **F2** menu. Move the cursor to any convenient point A and press ENTER. Move the cursor to a second point, B. Press ENTER and then CLEAR.

3. From the **F2** menu, choose **Segment**. Move the cursor to a point C well away from the line, and press ENTER. Move the cursor to any point D on the line. Press ENTER and then CLEAR.

4. Highlight **Measure** on the **F5** menu, press ▸, and choose **D.&Length** from the sub-menu. Move the cursor until segment CD flashes, and press ENTER and then CLEAR. Press ALPHA, move the measurement to a corner of the screen, and press ALPHA again.



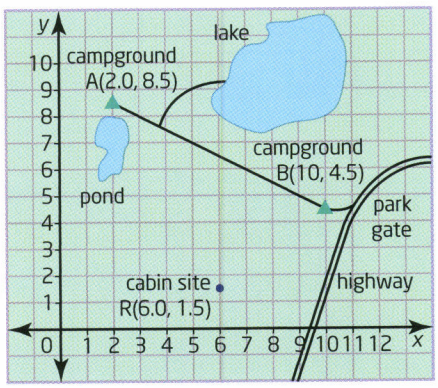
5. Move the cursor back to point D. Press ALPHA and use the cursor keys to slide point D along AB. Find the location that gives the shortest length for CD. To increase the precision displayed, move the cursor to the measurement and press +.

6. Estimate the measure of ∠ADC. To check your estimate, highlight **Measure** on the **F5** menu, press ▸, and choose **Angle**. Move the cursor toward point A until it flashes; then, press ENTER. Select point D and point C in the same way. Move the angle measurement to a convenient position and press ENTER. Was your estimate accurate?

7. **Reflect** What property does the shortest line segment from a point to a line have?

Example 1 Find the Shortest Route

A ranger cabin is to be built in a flat wooded area near the straight road that connects the two campgrounds in a park. A new side road will connect the cabin to the campground road. On the park map, the campgrounds have coordinates A(2.0, 8.5) and B(10.0, 4.5), while the site for the cabin is at R(6.0, 1.5). Each unit on the map grid represents 500 m.



a) Find the route that minimizes the cost and the number of trees that have to be cut down for the side road. Draw a diagram of this route.

b) Find the length of the side road, to the nearest tenth of a kilometre.

Solution

Since the area is level, the shortest route for the side road is the cheapest and easiest to build. The shortest route from the ranger cabin to the campground road is perpendicular to that road. To describe the route of the side road and to calculate its length, find the point where a perpendicular from R meets the line segment AB.

Understand the Problem

Use the coordinates of points A and B to calculate the slope of AB and find an equation for the line through A and B.

Choose a Strategy

The slope of the side road is the negative reciprocal of the slope of AB. Use this slope to determine an equation for the perpendicular line that passes through point R.

Use the equations for the two lines to find the point of intersection, D. Calculate the length of line segment RD from the coordinates of its endpoints. Then, use the map scale to find the length of the side road.

a) Calculate the slope of AB using the coordinates of the campgrounds, A(2.0, 8.5) and B(10.0, 4.5).

Carry Out the Strategy

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4.5 - 8.5}{10.0 - 2.0} \\ &= \frac{-4.0}{8.0} \\ &= -0.5 \end{aligned}$$

Since the slope of AB is -0.5, the slope of any line perpendicular to AB is $-\frac{1}{-0.5}$, or 2.

Perpendicular lines have slopes that are negative reciprocals of each other.

Now, find equations for AB and RD by substituting the slope and the coordinates of a point into $y = mx + b$.

For AB, use A(2.0, 8.5):

$$\begin{aligned} y &= mx + b \\ 8.5 &= -0.5(2.0) + b \\ 8.5 &= -1.0 + b \\ 9.5 &= b \end{aligned}$$

For RD, use R(6.0, 1.5):

$$\begin{aligned} y &= mx + b \\ 1.5 &= 2(6.0) + b \\ 1.5 &= 12.0 + b \\ -10.5 &= b \end{aligned}$$

I can use the coordinates of point B to check the equation for AB.

An equation for AB is $y = -0.5x + 9.5$ and an equation for RD is $y = 2x - 10.5$.

Use the substitution method to find the point of intersection of AB and RD. At the point of intersection, the y-coordinates of the two lines are equal, so

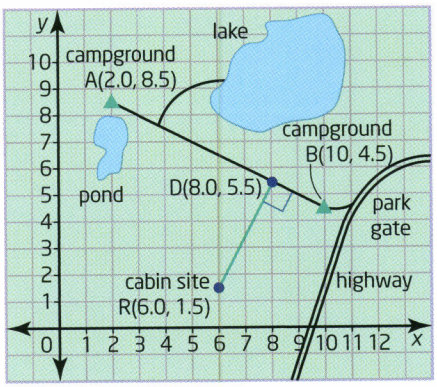
$$\begin{aligned} -0.5x + 9.5 &= 2x - 10.5 \\ 9.5 + 10.5 &= 2x + 0.5x \\ 20.0 &= 2.5x \\ \frac{20.0}{2.5} &= x \\ 8.0 &= x \end{aligned}$$

Substitute $x = 8.0$ into the equation for either line to find the y-coordinate of the point of intersection.

$$\begin{aligned} y &= -0.5x + 9.5 \\ &= -0.5(8.0) + 9.5 \\ &= -4.0 + 9.5 \\ &= 5.5 \end{aligned} \qquad \begin{aligned} y &= 2x - 10.5 \\ &= 2(8.0) - 10.5 \\ &= 16.0 - 10.5 \\ &= 5.5 \end{aligned}$$

The second equation shows that my calculation for the y-coordinate is correct.

The point of intersection of the two roads is D(8.0, 5.5). The best route for the side road to the ranger cabin is represented by the line segment joining (6.0, 1.5) to (8.0, 5.5).



b) To calculate the length of line segment RD, substitute the coordinates of its endpoints into the length formula.

$$\begin{aligned} \text{RD} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(8.0 - 6.0)^2 + (5.5 - 1.5)^2} \\ &= \sqrt{2.0^2 + 4.0^2} \\ &= \sqrt{4.0 + 16} \\ &= \sqrt{20} \\ &\doteq 4.5 \end{aligned}$$

I can use the graph to check that my answers are reasonable. It is hard to determine the exact coordinates or length from the graph.

Each unit on the map represents 500 m, so 4.5 units represents 4.5×500 m, or 2250 m.

The side road is 2.3 km long.

Reflect

Algebraic methods are particularly useful for situations that involve non-integer coordinates or lengths.

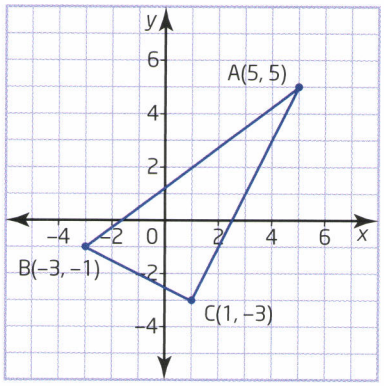
Example 2 Determine a Geometric Property Algebraically

The vertices of $\triangle ABC$ are A(5, 5), B(−3, −1), and C(1, −3). Determine whether $\triangle ABC$ is a right triangle.

Solution

Draw a diagram to help visualize the problem.

Understand the Problem



If $\triangle ABC$ is a right triangle, two of its sides are perpendicular to each other. Also, the Pythagorean theorem applies.

Choose a Strategy

Determine the slopes of the three sides of the triangle. Then, check if the product of any two of these slopes is −1.

Alternatively, calculate the lengths of all three sides, and check if the lengths satisfy the Pythagorean relation.

Method 1: Use Slopes

Carry Out the Strategy

Calculate the slope of each side of $\triangle ABC$.

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-1 - 5}{-3 - 5}$$
$$= \frac{-6}{-8}$$
$$= \frac{3}{4}$$

$$m_{BC} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-3 - (-1)}{1 - (-3)}$$
$$= \frac{-3 + 1}{1 + 3}$$
$$= \frac{-2}{4}$$
$$= -\frac{1}{2}$$

$$m_{AC} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-3 - 5}{1 - 5}$$
$$= \frac{-8}{-4}$$
$$= 2$$

Since $m_{AC} \times m_{BC} = -1$, $\angle ACB$ is a right angle. Therefore, $\triangle ABC$ is a right triangle.

Method 2: Use the Pythagorean Theorem

Find the length of each side of $\triangle ABC$ by substituting the coordinates of the vertices into the length formula.

$$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-3 - 5)^2 + (-1 - 5)^2} \\ &= \sqrt{(-8)^2 + (-6)^2} \\ &= \sqrt{64 + 36} \\ &= \sqrt{100} \\ &= 10 \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[1 - (-3)]^2 + [-3 - (-1)]^2} \\ &= \sqrt{(4)^2 + (-2)^2} \\ &= \sqrt{16 + 4} \\ &= \sqrt{20} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1 - 5)^2 + (-3 - 5)^2} \\ &= \sqrt{(-4)^2 + (-8)^2} \\ &= \sqrt{16 + 64} \\ &= \sqrt{80} \end{aligned}$$

Check whether the square of the longest side equals the sum of the squares of the two shorter sides:

$$AB^2 = 10^2 = 100$$

$$AC^2 = \left(\sqrt{80}\right)^2 = 80$$

$$BC^2 = \left(\sqrt{20}\right)^2 = 20$$

Squaring and taking the square root are opposite operations, so $(\sqrt{x})^2 = x$.

$$\begin{aligned} AC^2 + BC^2 &= 80 + 20 \\ &= 100 \\ &= AB^2 \end{aligned}$$

Since the Pythagorean relationship applies, $\triangle ABC$ is a right triangle.

Reflect

There is often more than one way to solve a problem using analytic geometry.

Example 3 Median to a Hypotenuse

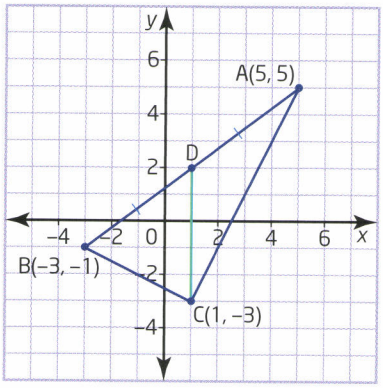
Show that the median from the right angle of the triangle in Example 2 is half as long as the hypotenuse.

Solution

Understand the Problem

You need to know the length of the hypotenuse and the length of the median from the right angle to the hypotenuse.

- A median joins a vertex of a triangle to the midpoint of the opposite side.
- From Example 2, you know that C is the vertex of the right angle and AB is the hypotenuse.
- Determine the coordinates of the midpoint of AB.
- Use these coordinates with the coordinates of vertex C to find the length of the median.
- Compare this length to the length of AB calculated in Example 2.



Carry Out the Strategy

Let point D be the midpoint of AB. Use the coordinates of vertices A and B to find the coordinates of D.

$$\begin{aligned}(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\\ &= \left(\frac{5 + (-3)}{2}, \frac{5 + (-1)}{2}\right)\\ &= \left(\frac{2}{2}, \frac{4}{2}\right)\\ &= (1, 2)\end{aligned}$$

The endpoints of the median from vertex C are C(1, −3) and D(1, 2). Substitute these coordinates into the length formula.

$$\begin{aligned}CD &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &= \sqrt{(1 - 1)^2 + [2 - (-3)]^2}\\ &= \sqrt{0^2 + 5^2}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

Since CD is vertical, I can also find its length from the difference of the *y*-coordinates of points C and D.

As shown in Example 2, substituting the coordinates A(5, 5) and B(−3, −1) into the length formula gives AB = 10. Since $CD = \frac{AB}{2}$, the median from the right angle is half as long as the hypotenuse.

Reflect

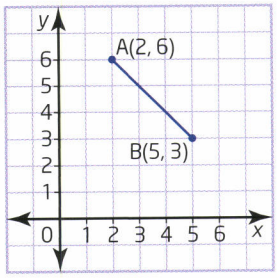
You could also use congruent triangles or geometry software to show the relationship between the length of the median and the length of the hypotenuse.

Key Concepts

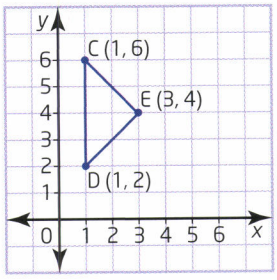
- You can use analytic geometry to determine properties of geometric shapes.
- These steps are helpful for solving multi-step problems:
  - Understand the Problem
  - Choose a Strategy
  - Carry Out the Strategy
  - Reflect
- A graph can be helpful for understanding a problem and for checking whether answers are reasonable.
- You can often use different methods to solve the same problem. Solving a problem in two different ways lets you check your calculations.

Communicate Your Understanding

C1 Describe how you would find an equation for the right bisector of line segment AB.



C2 Outline an algebraic method for showing that △CDE is an isosceles right triangle.

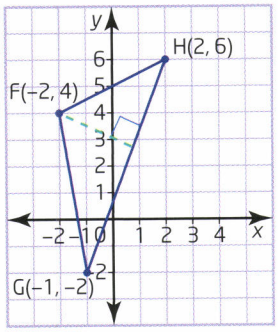


C3 Outline an algebraic method for finding the length of the **altitude** from vertex F of △FGH.

altitude

- height of a geometric shape

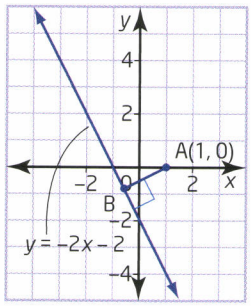
C4 Describe how you would find the coordinates of the point where the medians of △FGH intersect.



Practise

For help with question 1, see Example 1.

1. Find an equation for the line containing line segment AB.



For help with questions 2 and 3, see Example 2.

2. List two properties you could use to show that a triangle contains a right angle.

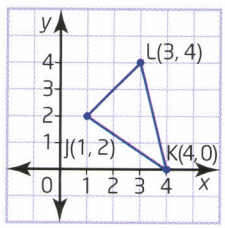
3. A triangle has vertices C(1, 4), D(−2, 2), and E(3, 1).

a) Draw △CDE.

b) Use analytic geometry to verify that ∠C is a right angle.

For help with question 4, see Example 3.

4. Find the length of the median from vertex K.

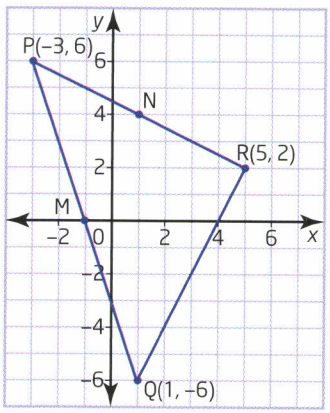


Connect and Apply

5. In △PQR, M is the midpoint of PQ and N is the midpoint of PR.

a) Show that MN is parallel to QR.

b) Show that MN is half the length of QR.



6. Determine whether the point T(2, −1) lies on the right bisector of the line segment with endpoints U(3, 5) and V(−3, −1). Explain your reasoning.

7. A quadrilateral has vertices O(0, 0), P(3, 5), Q(8, 6), and R(5, 1).

a) Determine whether OPQR is a parallelogram.

b) Describe how you could use geometry software to verify your answer to part a).

8. The endpoints of the diameter of a circle are M(−3, 5) and N(9, 7). Determine

a) the coordinates of the centre of the circle

b) the radius of the circle

9. Determine whether the triangle with vertices A(−3, 4), B(−1, −2), and C(3, 2) is isosceles.

10. Determine the shortest distance from the point (5, 2) to the line represented by $y = 2x + 1$. Use a diagram to check your answer.

11. Determine the shortest distance from the origin to the line represented by $y = \frac{1}{2}x - 2$.

12. Determine the shortest distance from the point D(5, 4) to the line represented by $3x + 5y - 4 = 0$.

13. Determine the shortest distance from the point E(1, −4) to the line through points F(−5, 2) and G(3, 4). Use a diagram to check your answer.

14. Determine the shortest distance from the point H(5, 2) to the line through points J(−6, 4) and K(−2, −4).

15. **Use Technology** Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify the solution to

a) Example 1

b) Example 2

c) Example 3

16. The points A(5, −3), B(−2, 4), and C(−1, 7) are three vertices of a parallelogram ABCD. Find the coordinates of vertex D. Check your answer by using a different method.

17. A triangle has vertices E(2, −2), F(−4, −4), and G(0, 4).

a) Determine an equation for the median from vertex E.

b) Determine the length of the median from vertex E.

18. **a)** Draw △DEF with vertices D(−1, 6), E(4, 3), and F(0, −4). Then, draw the altitude from vertex D.

b) Find an equation for the altitude from vertex D.

19. **Use Technology** Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 18. Describe the method you used.

20. A quadrilateral has vertices P(−5, 4), Q(−2, 8), R(6, 2), and S(3, −2).

a) Show that the quadrilateral is a rectangle.

b) Determine the length of each diagonal.

c) Determine the midpoint of each diagonal.

d) What can you conclude about the diagonals of PQRS?

21. A triangle has vertices J(−2, 0), K(4, −3), and L(8, 8).

a) Find an equation for the altitude from vertex L to side JK.

b) Find the length of the altitude.

c) Find the area of △JKL.

22. **Use Technology** Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 20. Describe the method you used.

23. A cable company is connecting a new customer to its cable network. On a site plan, the customer's house has coordinates H(7, 17). The equation $y = \frac{1}{2}x + 4$ represents the existing trunk cable. The cable company wants to keep the branch to the customer's house as short as possible.

a) Where should the cable company make the connection to the trunk cable?

b) How long will the branch connection be if each unit on the grid of the site plan represents 10 m?

24. Dylan and Indira are hiking on the Caledon Hills section of the Bruce Trail. They have reached the point that has coordinates (6, 8) on their map of the trail. They want to hike out to the straight section of Hockley Road that joins points (4, 7) and (6, 5).

a) At what point will they reach Hockley Road if they take the shortest possible route?

b) Explain why the shortest route might not be the best route.

25. A utility company is running new power lines to two cottages. On a site plan, the cottages have coordinates A(6, 7) and B(13, 6) and the closest transformer is at T(13, 14). The utility will run a line straight from the transformer to one of the cottages and then connect the other cottage to that line using the shortest possible route.

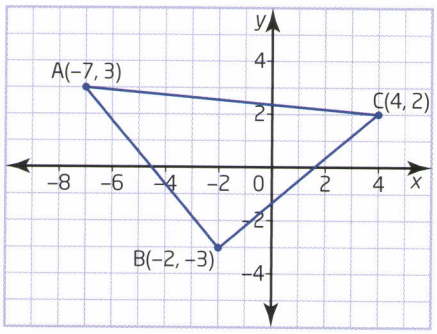
a) Draw a diagram on a grid to show the two possible ways to run the power lines.

b) Determine which route will require the least cable.

26. **Use Technology** Use geometry software to verify your answer to question 25. Describe the method you used.

Achievement Check

27. a) Determine an equation for the median from vertex A of △ABC.



b) Determine an equation for the right bisector of BC.

c) Are the equations in parts a) and b) the same?

d) What property must a triangle have if the median to one of its sides coincides with the right bisector of that side?

Extend

28. a) Draw the triangle with vertices A(2, 1), B(4, −1), and C(−2, −5). Then, construct the median from each vertex.

b) Verify algebraically that the three medians intersect at a single point, the **centroid** of △ABC.

29. Use Technology Use *The Geometer's Sketchpad®* or Cabri® Jr. to determine whether the median to the hypotenuse of a right triangle is always half as long as the hypotenuse. Describe your method and your findings.

30. In three dimensions, the location of a point can be represented by the ordered triple (x, y, z).

a) Find the length of the line segment with endpoints P(2, 3, 1) and Q(6, 6, 5).

b) Write a formula for the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

31. The municipal sewer line runs straight through a new subdivision from point A(20, 20) to point B(80, 60) on a survey map. Houses at C(30, 70) and D(85, 20) need connections to this sewer line. The developer calculates that connecting to the sewer line at points E(50, 40) and F(65, 50) will minimize digging and the length of pipe required.

a) Verify that the developer has found the shortest route from each house to the sewer line.

b) To the nearest metre, what length of pipe is needed for the two connections if the intervals between grid lines on the survey each represent 2 m?

c) The excavation contractor suggests digging a straight trench between the two houses and connecting to the sewer line at the point where the trench meets it. Find the coordinates of this point.

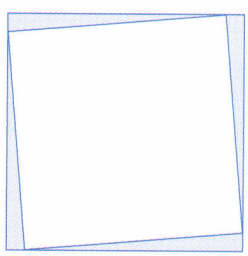
d) Should the developer use the contractor's suggestion? Justify your answer.

32. Math Contest In factorial notation, $n!$ represents the product $n(n-1)(n-2)\ldots(3)(2)(1)$.
If $x! = 3!5!7!$, the value of x is

A 10 **B** 8 **C** 11 **D** 9 **E** 12

33. Math Contest The perimeter of the smaller square is 96 cm and the shaded area is 100 cm^2. The perimeter of the larger square is

A 40 cm **B** 72 cm **C** 104 cm
D 144 cm **E** 400 cm



2.4 Equation for a Circle

A licence is not required for portable two-way radios with a power of up to 2 W operating on the General Mobile Radio Service (GMRS) frequencies in Canada. GMRS radios are similar to Family Radio Service (FRS) radios, which are limited to 0.5 W and use different frequencies. Some radios are hybrids that can operate on both the GMRS and the FRS frequencies.

Investigate

How can you find an equation for a circle?

Tools

- grid paper
- compasses

Near his home, Trevor's GMRS radios have a range of about 5 km.

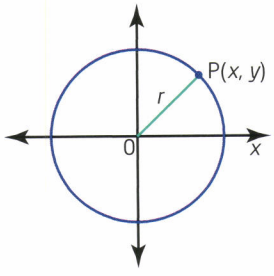
Method 1: Use Pencil and Paper

1. Draw a circle to represent the range of Trevor's radios. Let the origin represent Trevor's position.

2. Label the x- and y-intercepts of your circle. What do these intercepts have in common?

3. Find four other points on the circle that have integer coordinates. Label these points A, B, C, and D, and mark their coordinates on your drawing. Use the distance formula to verify that each of these points is exactly 5 units from the origin.

4. Mark a point P(x, y) anywhere on the circle. Construct a right triangle with OP as the hypotenuse and the rise and the run of OP as the other two sides.



5. Write an equation relating the length of OP to the length of the other two sides of the right triangle. Substitute OP = 5 into the equation.

6. Verify that the coordinates of points A, B, C, and D satisfy the equation in step 5.

7. **Reflect** Will the coordinates of every other point on the circle also satisfy the equation? Explain your reasoning.

8. Away from built-up areas, Trevor finds that his GMRS radios have a range of about 7 km. Add a circle to your drawing to represent this larger range.

9. Write an equation for the larger circle.

10. **Reflect** Write an equation for the circle with centre (0, 0) and radius r. Then, use this equation to write an expression for the radius.

Tools
- computer with *The Geometer's Sketchpad®*

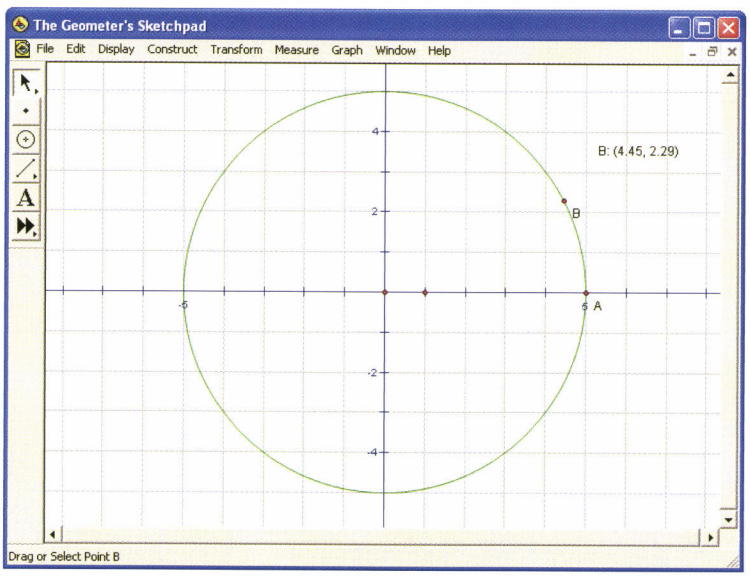
Method 2: Use *The Geometer's Sketchpad®*

1. Choose **Show Grid** from the **Graph** menu.

2. Construct a circle to represent the range of Trevor's radios. Let the origin represent Trevor's position. Use the **Compass Tool** to construct a circle with its centre at the origin and a radius of 5 units.

3. Label the x- and y-intercepts of the circle. What do these points have in common?

4. Use the **Point Tool** to construct a point on the circle. Select the point and choose **Coordinates** from the **Measure** menu. Then, drag the point around the circumference of the circle to find four other points that have integer coordinates. Construct these points and label them with their coordinates.



5. Verify that each of the points is 5 units from the origin. Select a point and the origin, and choose **Distance** from the **Measure** menu. Use the same method to measure the distance from the origin to each of the other points.

6. Construct a point anywhere on the circle. Label the point P. Construct a right triangle with OP as the hypotenuse and the rise and the run of OP as the other two sides.

7. Write an equation relating the length of OP to the length of the other two sides of the right triangle. Substitute OP = 5 into the equation.

8. Verify that the coordinates of the points in step 4 satisfy the equation in step 7.

9. **Reflect** Will the coordinates of every other point on the circle also satisfy the equation? Explain your reasoning.

10. Away from built-up areas, Trevor finds that his GMRS radios have a range of about 7 km. Add a circle to your drawing to represent this larger range.

11. Write an equation for the larger circle.

12. **Reflect** Write an equation for the circle with centre (0, 0) and radius r. Then, use this equation to write an expression for the radius.

Example 1 Equation for a Circle

Find an equation for the circle with centre (0, 0) and radius 4.

Solution

The distance from the origin to any point P(x, y) on the circle is the length of the radius. So,

OP = 4

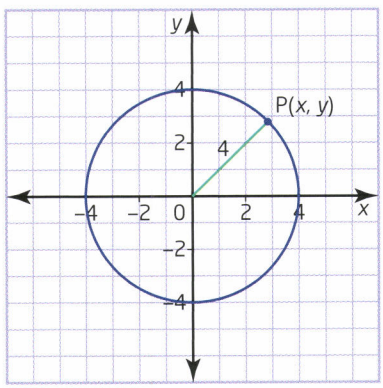
The distance formula also gives an expression for the length of OP:

$$\begin{aligned} \text{OP} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(x - 0)^2 + (y - 0)^2} \\ &= \sqrt{x^2 + y^2} \end{aligned}$$

Therefore,

$$\begin{aligned} \sqrt{x^2 + y^2} &= 4 \\ x^2 + y^2 &= 16 \end{aligned}$$

An equation for the circle is $x^2 + y^2 = 16$.



Example 2 Determine Whether a Point Lies Within a Circle

a) Determine an equation and the radius for the circle that has its centre at the origin and passes through the point A(6, −8).

b) Is the point B(−5, 9) inside this circle?

Solution

a) An equation for a circle centred at the origin has the form $x^2 + y^2 = r^2$.

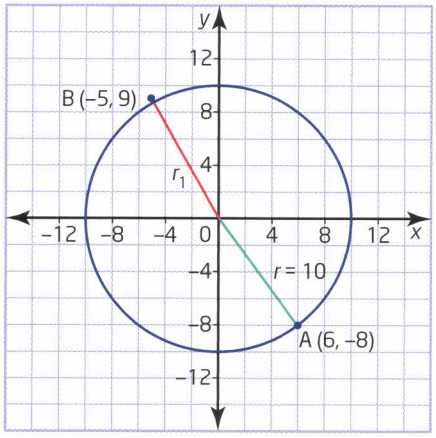
Substitute the coordinates of the point (6, −8) into the equation for the circle.

The point (6, −8) lies on this circle, so the coordinates of the point must satisfy the equation of the circle.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ 6^2 + (-8)^2 &= r^2 \\ 36 + 64 &= r^2 \\ 100 &= r^2 \\ \sqrt{100} &= \sqrt{r^2} \\ 10 &= r \end{aligned}$$

An equation for the circle is $x^2 + y^2 = 100$, and the radius of the circle is 10.

b) Consider a circle with its centre at the origin and with point B(−5, 9) on the circumference. Let r_1 be the radius of this circle. To find the length of the radius, substitute the coordinates of point B into the formula for the radius of a circle centred at the origin.



$$\begin{aligned} r_1 &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-5)^2 + 9^2} \\ &= \sqrt{25 + 81} \\ &= \sqrt{106} \\ &\doteq 10.3 \end{aligned}$$

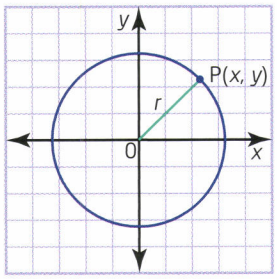
Since $r_1 > 10$, point B lies outside the circle defined by $x^2 + y^2 = 100$.

If $r_1 > r$, then $r_1^2 > r^2$. So, the inequality $x^2 + y^2 > r^2$ defines the region *outside* the circle with centre (0, 0) and radius r.

Reflect

Key Concepts

- An equation for the circle with centre at the origin and radius r is $x^2 + y^2 = r^2$.
- The radius of a circle centred at the origin is $r = \sqrt{x^2 + y^2}$.



Communicate Your Understanding

C1 Outline how you would find an equation for the circle centred at the origin with a radius of 12.

C2 Describe how you would determine whether the point (3, 5) lies on the circle defined by $x^2 + y^2 = 35$.

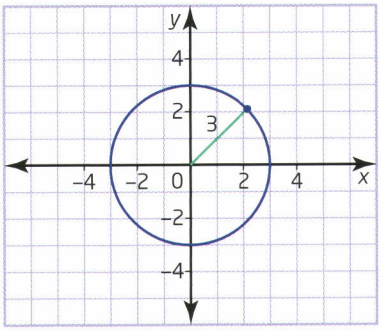
C3 Explain how you would determine whether the point (8, 8) lies inside the circle defined by $x^2 + y^2 = 100$.

Practise

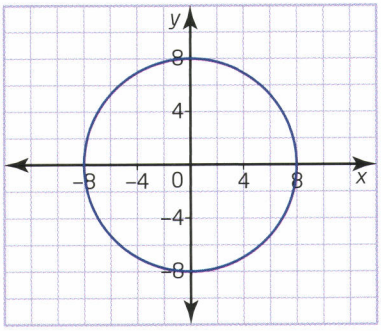
For help with question 1, see Example 1.

1. Determine an equation for each circle.

a)



b)



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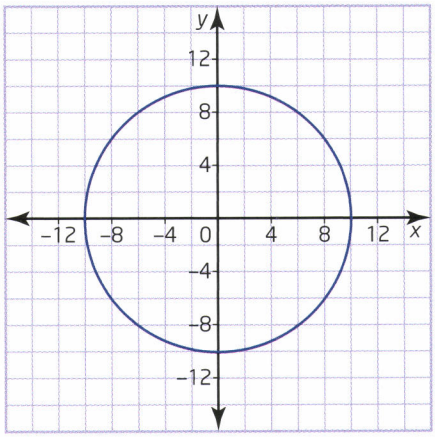

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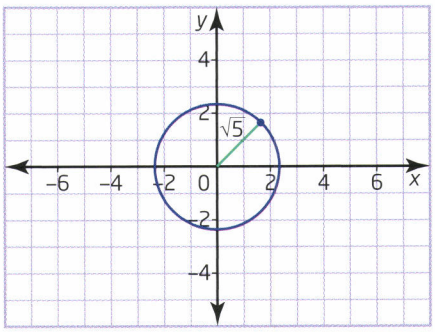

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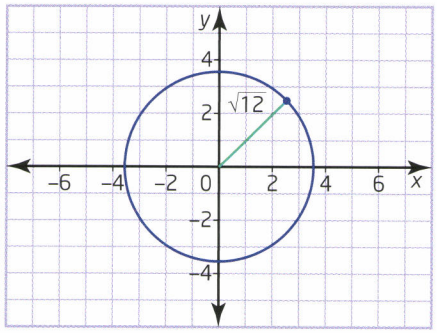
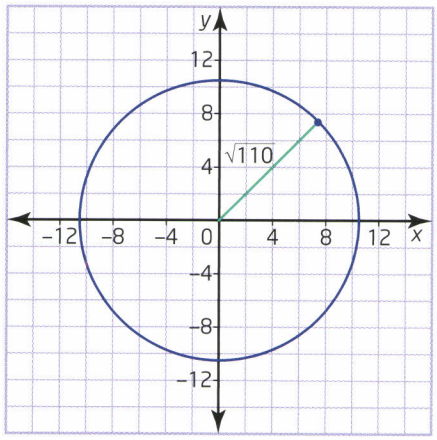


f)



For help with questions 2 to 4, see Example 2.

2. For each equation, state the radius of the corresponding circle and give the coordinates of one point on the circle.

a) $x^2 + y^2 = 36$ **b)** $x^2 + y^2 = 144$

c) $x^2 + y^2 = 20$ **d)** $x^2 + y^2 = 50$

e) $x^2 + y^2 = 1.69$

3. For each point, find an equation for the circle that is centred at the origin and passes through the point. Then, check your answer by graphing the circle and plotting the point.

a) $(-4, 3)$ **b)** $(5, 2)$

c) $(-3, -6)$ **d)** $(-7, 12)$

4. Determine whether each point is on, inside, or outside the circle defined by $x^2 + y^2 = 34$.

a) $(5, -3)$ **b)** $(4, 4)$

c) $(-6, 0)$ **d)** $(-3, -5)$

e) $(2, -6)$ **f)** $(\sqrt{34}, 0)$

Connect and Apply

5. A satellite orbits Earth on a circular path with equation $x^2 + y^2 = 1.44 \times 10^8$, with distances measured in kilometres. Another satellite orbiting in the same plane passes through the point (8000, 9800). Is this satellite inside the orbit of the first one?

6. Determine an equation for the circle that has a diameter with endpoints A$(-4, 3)$ and B$(4, -3)$.

7. The point $(a, 8)$ lies on the circle defined by $x^2 + y^2 = 100$.

a) Explain why there are two possible values for a. Find these values.

b) Use a graph to check that the points corresponding to both values for a are on the circle.

8. A farmer is building a circular corral to hold livestock. With distances measured in metres, the shape of the corral is modelled by the equation $x^2 + y^2 = 64$.

a) Find the length of fencing required for this corral.

b) Find the area of the corral.

9. a) Graph the circle defined by $x^2 + y^2 = 100$.

b) Verify algebraically that the points P$(-8, 6)$ and Q$(6, 8)$ are both on the circle.

c) Find an equation for the right bisector of the **chord** PQ.

d) Verify that the right bisector in part c) passes through the centre of the circle.

e) Do you think that the right bisector of any chord of the circle passes through the centre of the circle? Explain your reasoning.

10. **a)** Graph the circle defined by $x^2 + y^2 = 40$.

b) Verify algebraically that the line segment joining R$(-6, 2)$ and S$(2, -6)$ is a chord of this circle.

c) Determine an equation for the line joining the centre O to the midpoint of this chord.

d) Verify that this line is perpendicular to the chord.

11. **a)** Graph the circle defined by $x^2 + y^2 = 41$.

b) Verify algebraically that the line segment joining U$(-4, 5)$ and V$(-5, -4)$ is a chord of this circle.

c) Determine an equation for the line that passes through the origin and is perpendicular to the chord UV.

d) Verify that this line passes through the midpoint of the chord.

12. **Use Technology** Use geometry software to determine whether the right bisector of any chord of a circle passes through the centre of the circle. Describe the method you used and your findings.

13. **a)** Graph the circle defined by $x^2 + y^2 = 25$.

b) Verify algebraically that the point A$(-3, -4)$ lies on the circle.

c) Construct the line segment AO.

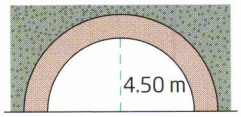
d) Draw the line through A that is perpendicular to AO. This perpendicular line is a **tangent of a circle**.

e) Determine an equation for the tangent in part d).

f) Explain why A is the only point that is on both the circle and the tangent.

14. Brandon has three close friends who live in different parts of the city. Brandon wants to meet them for lunch at a restaurant that is roughly equidistant from their homes. How could Brandon use his knowledge of circles to help find a suitable restaurant? Explain your reasoning.

15. As part of the North American Free Trade Agreement (NAFTA), Canada, the United States, and Mexico are developing joint standards for highway trucks. One standard specifies a maximum width of 2.60 m and a maximum height of 4.15 m. Will a truck of this size fit through a semicircular tunnel with a maximum height of 4.50 m?



16. Lei is designing a construction set for small children. The set includes cylindrical cups with radii of 5 cm, 6 cm, and 7 cm as well as rectangular blocks that measure 7 cm by 8 cm by 9 cm. Will the blocks fit inside all of the cups?

17. A ship drops its anchor into the water and creates a circular ripple. The radius of this ripple increases at a rate of 50 cm/s.

a) Find an equation for the circle 10 s after the anchor is dropped.

b) A small rowboat is 50 m east and 75 m north of the point where the anchor was dropped. How long does the ripple take to reach the rowboat?

c) Describe any assumptions you made for your answers to parts a) and b).

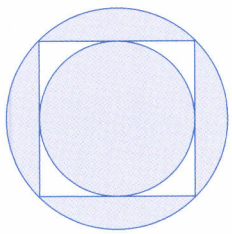
Extend

18. Describe the region defined by each inequality. Then, draw and label a diagram of the three regions.

a) $x^2 + y^2 < 25$ **b)** $x^2 + y^2 > 49$

c) $25 < x^2 + y^2 < 49$

19. An equation for the small circle in this design is $x^2 + y^2 = 4$. Determine an equation for the larger circle.



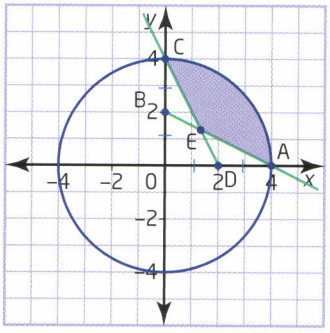
20. **a)** Write an equation for the circle in the diagram below.

b) Write the coordinates of points B and D.

c) Determine equations for the lines containing line segments AB and CD.

d) Determine the coordinates of point E.

e) Determine the area of the shaded portion of the circle.



21. A boat moving east at 2 m/s creates circular waves that travel outward from the boat at a speed of 1 m/s.

a) How far does the wave produced by the boat travel in the time it takes the boat to move 10 m?

b) Plot the waves when the boat has moved 10 m. Draw circles at 1-m intervals from the origin to (8, 0).

c) Describe the pattern formed by the points of intersection of the waves. What does this pattern represent?

22. Find an equation for the circle centred at (4, 3) with a radius of 5.

23. Use a counterexample to show that this statement is false: "Every circle with a radius greater than 1 has at least one point with integer coordinates."

24. **a)** Draw the triangle with vertices Q(−2, 0), R(2, 8), and S(7, 3). Then, construct the right bisector of each side.

b) Verify algebraically that the three right bisectors intersect at a single point, the **circumcentre** of △QRS.

c) Find the distance from each vertex of △QRS to the circumcentre.

d) Describe the circle that passes through the vertices of △QRS.

e) Describe how you would use geometry software to answer parts a) to d).

25. **Math Contest** Find the radius of the circle represented by the equation $kx^2 + ky^2 = r^2$, where $k > 0$.

26. **Math Contest** Given that $a > 0$ and $b > 0$, describe the graph of the equation $ax^2 + by^2 = r^2$ if

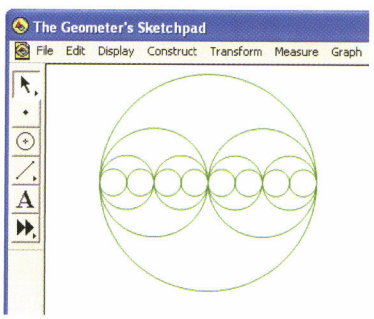
a) $a < b$

b) $a > b$

Making Connections

You can use *The Geometer's Sketchpad®* to generate a variety of fractals. One of the easiest is a nest of circles.

- From the **Edit** menu, choose **Preferences**. Click on the **Text** tab. Ensure that **For All New Points** is checked.
- Construct a horizontal line segment AB. Select AB and choose **Midpoint** from the **Construct** menu. Construct a circle with centre C and radius CB. Hide line segment AB.
- Select points A and B, in that order. Choose **Iterate** from the **Transform** menu. Map A onto A and B onto C. Choose **Add New Map** from the **Structure** menu. Map A onto C and B onto B.
- Press the + key to add iterations and the − key to remove iterations. When you are satisfied with the image, click on **Iterate**.
- Clean up the sketch by hiding all visible points.

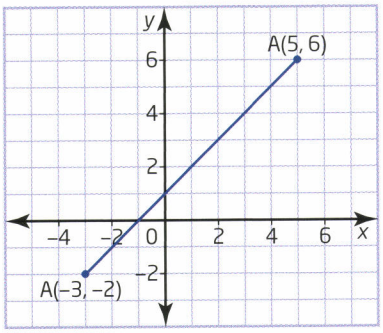


Chapter 2 Review

2.1 Midpoint of a Line Segment, pages 56–69

1. Find the midpoint of each line segment.

a)



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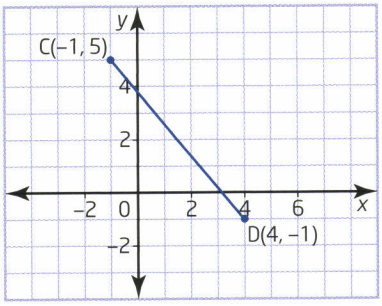

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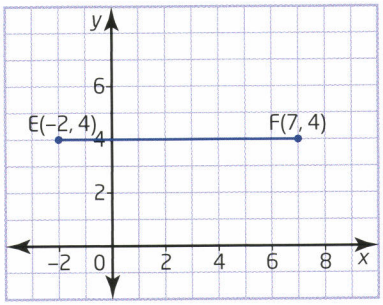

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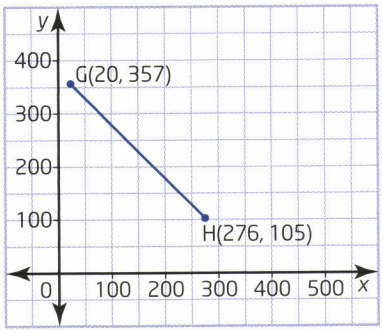


2. a) Determine the midpoint of the line segment with endpoints J(3, −5) and K(−5, −6).

b) Determine the midpoint of the line segment with endpoints L(4, 8) and N(4, −2).

3. a) Draw the triangle with vertices P(−2, 5), Q(6, 5), and R(2, −7).

b) Determine the midpoint of each side of the triangle algebraically.

c) Join the midpoints to form a smaller triangle. Compare this triangle to the original triangle.

4. a) Draw the triangle with vertices T(−8, 6), U(2, 10), and V(4, −4).

b) Draw the median from vertex U. Then, find an equation for this median.

c) Draw the altitude from vertex T. Then, find an equation for this altitude.

d) Draw the right bisector of TU. Then, find an equation for this right bisector.

5. The midpoints of the sides of △ABC are D(4, 1), E(−2, 3), and F(1, −4).

a) Plot the midpoints. Use this plot to estimate the coordinates of the vertices of △ABC.

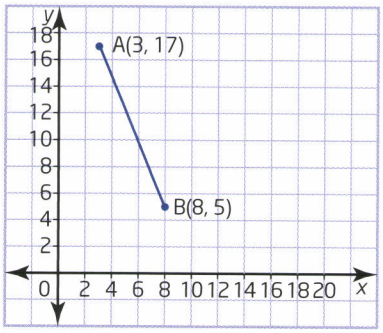
b) Use analytic geometry to calculate the coordinates of the vertices of △ABC.

c) Describe how to use geometry software to find the coordinates of the vertices.

2.2 Length of a Line Segment, pages 70–79

6. Find the length of each line segment.

a)



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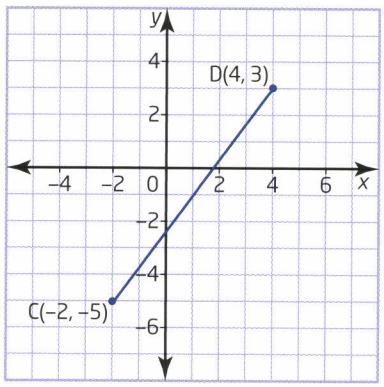

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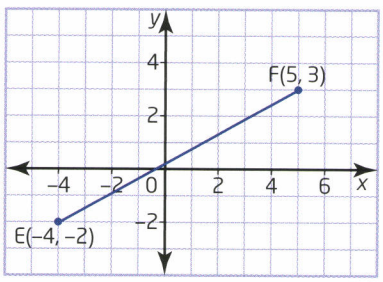
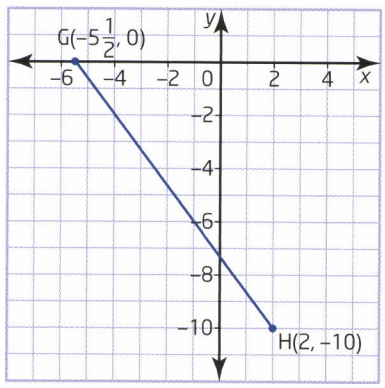


d)



7. Determine the length of the line segment defined by each pair of points.

a) J(4, 8) and K(4, −2)

b) M(−3, −12) and N(−15, −7)

c) P(−3, −2) and Q(5, 6)

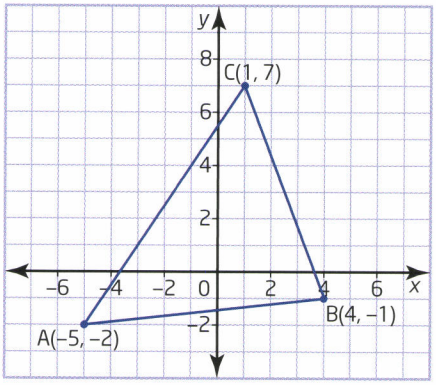
d) R(−1, 5) and S(4, −1)

e) T(−2, 4) and U(7, 4)

f) V(3, −5) and W(−5, −6)

8. a) Determine the length of the median from vertex A of △ABC.

b) Determine the perimeter of the triangle.



9. a) Draw the triangle with vertices D(5, 25), E(−10, 1), and F(3, −10).

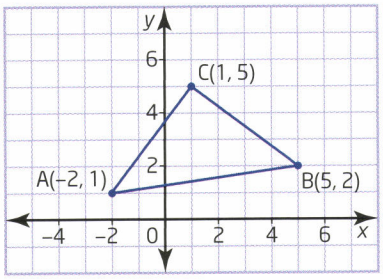
b) Use analytic geometry to classify △DEF.

c) Determine the area of △DEF.

d) Describe how to use geometry software to answer part c).

2.3 Apply Slope, Midpoint, and Length Formulas, pages 80–91

10. Show that this triangle is isosceles.



11. A triangle has vertices D(−2, 7), E(−4, 2), and F(6, −2).

a) Show algebraically that this triangle is a right triangle.

b) Find the midpoint of the hypotenuse.

c) Show that this midpoint is equidistant from each of the vertices.

12. A map shows a main gas pipeline running straight from A(45, 60) to B(65, 40).

a) How long is the section of pipeline from A to B if each unit on the map grid represents 1 km?

b) A branch pipeline runs perpendicular to the main pipeline and meets it at a point halfway between A and B. Find the coordinates of this point.

c) Is the point C(63, 54) on the branch pipeline? Explain your reasoning.

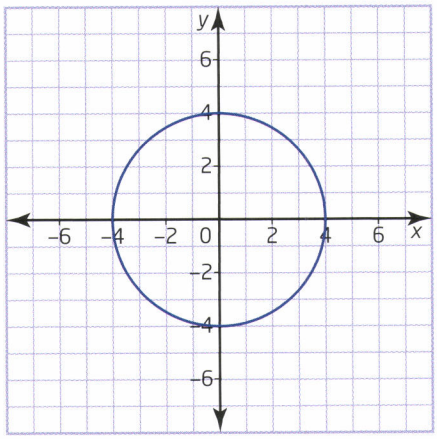
d) What is the shortest route for connecting point C to the main pipeline? Explain.

13. Find the shortest distance from the origin to the line defined by $y = 3x - 10$.

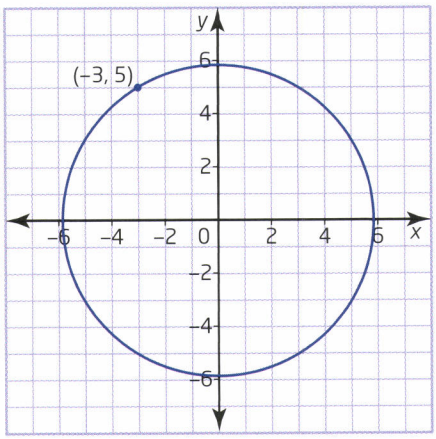
2.4 Equation for a Circle, pages 92–99

14. Determine an equation for each circle.

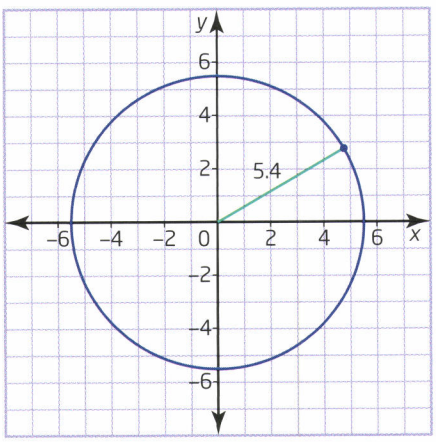
a)



b)



c)



15. Find an equation for the circle that is centred on the origin and

a) has a radius of $4\frac{1}{2}$

b) has a diameter of 14

c) has a radius of $\sqrt{12}$

d) passes through the point (4, 7)

16. a) Determine whether the point A(−2, −6) lies on the circle defined by $x^2 + y^2 = 40$.

b) Find an equation for the radius from the origin O to point A.

c) Find an equation for the line that passes through A and is perpendicular to OA.

d) Use a graph to check your answers to parts a), b), and c).

e) Explain why A is the only point on the line that also lies on the circle.

17. a) Show that the line segment joining A(−3, 1) and B(1, 3) is a chord of the circle defined by $x^2 + y^2 = 10$.

b) Determine an equation for the right bisector of the chord AB.

c) Show that the line in part b) passes through the centre of the circle.

18. A communication tower can send and receive signals from cell phones up to 20 km away. A cell phone user is 15 km east and 13 km south of the tower. Is this user able to receive a signal from the tower?

Chapter Problem Wrap-Up

Many natural systems, such as river deltas and tree branches, have patterns that look like fractals. Parts within a natural pattern usually vary somewhat, so they are not exactly similar. Nonetheless, fractals often provided the first reasonably accurate mathematical model for such complex systems.

Beginning with a discovery by the meteorologist Edward Lorentz in 1961, scientists and mathematicians found that fractal models revealed surprising patterns in data that had appeared to be completely random. The mathematics that describes this hidden order in complex systems is called chaos theory. Applications of chaos theory include models of ocean currents, population growth, commodity prices, and blood flow in veins and arteries.

a) Research fractals at a library or on the Internet. Describe how to create a fractal different from the ones mentioned in this chapter. Choose a fractal you learned about during your research or create one of your own. Explain how midpoints, lengths, or equations for circles are used to generate the fractal.

b) Make a poster of the fractal you described in part a). Draw the fractal by hand or by using technology. Describe any special features of the fractal.

c) Describe a specific application of fractals, such as a scientific model or computer-generated imaging (CGI).

Chapter 2 Practice Test

For questions 1 to 3, choose the correct answer.

1. The midpoint of the line segment with endpoints A(−3, −3) and B(1, 5) is at

A (−2, 2) **B** (−4, −8)

C (−1, 1) **D** (1, −1)

2. The length of the line segment with endpoints C(−5, 2) and D(1, −4) is

A $\sqrt{20}$ **B** $\sqrt{24}$

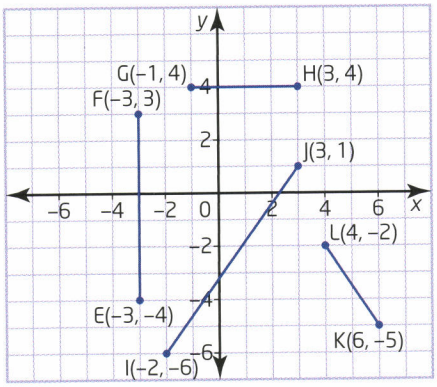
C $\sqrt{72}$ **D** $\sqrt{80}$

3. An equation for the circle with centre (0, 0) and radius 4 is

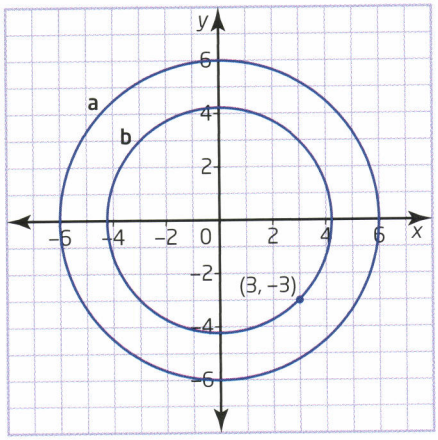
A $x^2 + y^2 = 2$ **B** $x^2 + y^2 = 4$

C $x^2 + y^2 = 8$ **D** $x^2 + y^2 = 16$

4. Determine the midpoint coordinates and the length of each line segment.

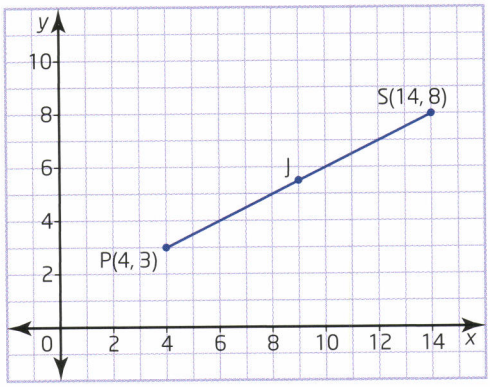


5. Write an equation for each circle.



6. Rachel makes the following statement: "Since point A is the same distance from both B and C, A is the midpoint of BC." Is Rachel correct? Explain your reasoning.

7. Jason lives exactly halfway between the primary and secondary schools in his neighbourhood. The intervals between the grid lines represent 1 km.



a) How far apart are the schools?

b) Determine the coordinates of Jason's home.

c) What other locations are equidistant from the two schools? Explain your reasoning.

d) Determine an equation that represents all locations that are equidistant from the two schools.

8. a) Plot the triangle with vertices A(−2, 1), B(2, −1), and C(0, 5).

b) Determine the lengths of the sides of the triangle.

c) Classify △ABC. Explain your reasoning.

d) Find the area of △ABC.

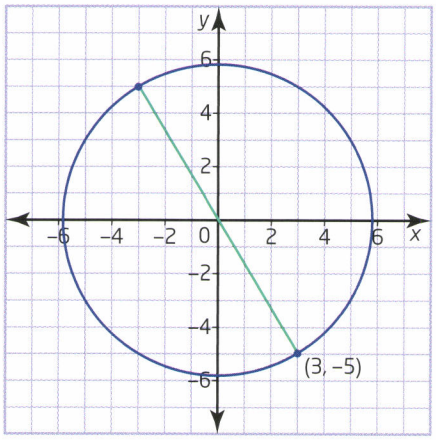
e) Describe how you could use geometry software to verify your answers to parts b), c), and d).

9. a) Plot the triangle with vertices P(3, 4), Q(−5, 2), and R(1, −4). Then, draw the median from vertex R.

b) Determine an equation for this median.

c) Is this median also an altitude for this triangle? Justify your answer.

10. a) Determine the coordinates of the other endpoint of the diameter shown.



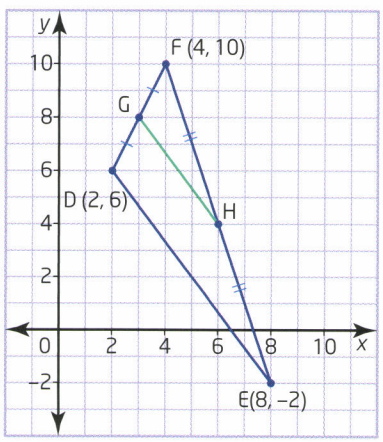
b) Does any other point on the circle have an x-coordinate of 3? Justify your answer.

c) Determine an equation for the circle.

d) Explain how you can use the equation of the circle to verify your answer to part b).

e) Determine the coordinates of four more points on the circle.

11. a) Determine the coordinates of the midpoints G and H.



b) Verify that GH is parallel to DE.

c) Show that GH is exactly half the length of DE.

12. a) Show that the triangle with vertices U(4, 3), V(0, −5), and W(−4, −3) is a right triangle.

b) Verify that the median from the right angle to the hypotenuse is half as long as the hypotenuse.

c) Find an equation for the circle that passes through the vertices of △UVW.

d) Describe how you could use geometry software to verify your answers to parts a), b), and c).

13. Scott, Arif, and Diane run a small delivery company. For their business, they use licensed two-way radios with a 20-km range. Scott is at their office, which they have marked as the origin on their map of the town. The grid lines on the map are spaced 1 km apart. Arif is dropping off a package at (−8, 16) while Diane is making a pick-up at (4, 20).

a) Draw a diagram to represent the reception range for the radio at the office.

b) Find an equation that describes the boundary of this area.

c) Are Arif and Diane both within range of the radio at the office? Justify your answer.

d) Are Arif and Diane within radio range of each other? Justify your answer.

Achievement Check

14. A(9, 5) and B(5, −9) are two points on a circle centred at the origin.

a) Determine an equation for the circle.

b) Determine the midpoint, C, of chord AB.

c) Show that the right bisector of chord AB passes through the centre of the circle.

d) Give a different solution for part c).